II nd year - III rd semester

course code : 7MMA3C1

core course - IX - complex analysis

C - 1

Unit : I

concept of analytic functions - Elementary theory of power series - conformability - Linear transformations.

Unit : II

complex integration - cauchy integral formula.

Unit : III

Local properties of analytic functio[ns]

Unit : IV

calculus of Residues.

Unit : V

power series expansions - canonical products - Jensen's formula.

Text book :

Lars V. Ahlfors, complex analysis 3rd edition, McGraw Hill international book company 1979.

chapter II : (sections 1, 2).
chapter III : (" 2, 3)
" IV : (" 1, 2, 3 & 5).
" V : (" 1.1, 1.2, 1.3, 2.1, 2.2, 2.3, 3.3)

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Books for supplementary Reading and references:

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1. S. ponnusamy, foundations of complex analysis, Narosa publication house. New Delhi, 2004.

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Unit - I.

C - (3)

P - (3) Defn: (Limit and continuity).

Limit:

The function $f(x)$ is said to have limit A as $x \to$ to a if

$$\lim_{x \to a} f(x) = A.$$

continuity:

For every $\epsilon > 0$ there exists a number $\delta > 0$. with the property that

$|f(x) - A| < \epsilon$ for all values of x

Such that

$|x - a| < \delta$ and $x \neq a$.

concept of analytic function: (or) holomorphic function:

The sum and the product of two analytic functions are again analytic the same is true of the quotient

$f(z) / g(z)$.

Of two analytic functions provided that $G(z)$ does not vanish $G(z) \neq 0$ ($\frac{16}{16}$, $\frac{16}{16}$, $\frac{16}{16}$ ...) In the general case it is necessary to exclude the points of at which $G(z) = 0$.

The defn of the derivatives can be written in the form.

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

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P-(1) As a first consequence $f(z)$ is necessarily continuous of the fn

$\Rightarrow f(z+h) - f(z) = \frac{h(f(z+h) - f(z))}{h}$

C-(4)

$\Rightarrow \lim_{h \to 0} (f(z+h) - f(z)) = 0$

$f(z) + f(h) - f(z) = 0$

$\Rightarrow f'(z) = 0$

$\Rightarrow f(z) = u(z) + iv(z)$

Thus $u(z)$ and $v(z)$ are continuous.

The limit of the different quotient must be the same of the way in which h aproaches 0.

$\frac{\partial f}{\partial x}$ ... $\frac{\partial y}{\partial x}$ ... $\frac{\partial v}{\partial x}$

lly

we substitude purely imaginary values ik for h we obtain.

$$f'(z) = \lim_{k \to 0} \frac{f(z+ik) - f(z)}{ik}$$

$$= -i \frac{\partial f}{\partial y} \quad \left(\because \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)$$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

it is follows that $f(z)$ must satisfy the partial differential

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$$\boxed{\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}}$$

which resolves into the real eqn.

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$ C. R. equation → (A).

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

for the quantity $|f'(z)|^2$

we have

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2$$

$$= \boxed{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}$$

The last expression shows that $|f'(z)|^2$ is the Jacobian of u and v with respect to x and y.

Let u and v will have continuous partial derivatives of all orders.

$$\boxed{\begin{aligned}\Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.\\ \Delta v &= \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0.\end{aligned}}$$

A function u which satisfies the Laplace equation $\Delta u = 0$ is said to be harmonic.

P-6 If Two harmonic function u and v satisfies the cauchy Riemann equation in (A) then v is said to be conjugate harmonic function of u. C-(6)

If u and v have continuous first-order partial derivatives it is proved in calculus under exactly these regularity conditions.

that we can write

$$u(x+h, y+k) - u(x,y) = \frac{\partial u}{\partial x} h + \frac{\partial u}{\partial y} k + \epsilon_1$$

$$v(x+h, y+k) - v(x,y) = \frac{\partial v}{\partial x} h + \frac{\partial v}{\partial y} k + \epsilon_2$$

where the remainders ϵ_1, ϵ_2 tend to zero more rapidly than $h+ik$ in the sense that $\epsilon_1/(h+ik) \to 0$ and $\epsilon_2/(h+ik) \to$ for $h+ik \to 0$ with the notation

$$f(z) = u(x,y) + iv(x,y).$$

we obtain by virtue of the relation (6)

$$f(z+h+ik) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(h+ik) + \epsilon_1 + i\epsilon_2$$

and hence

$$\lim_{h+ik \to 0} \frac{f(z+h+ik) - f(z)}{h+ik} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

we conclude that $f(z)$ is analytic

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(1) If $u(x,y)$ and $v(x,y)$ have continuous first order partial derivatives which satisfy the C-R differential can then $f(z) = u(z) + iv(z)$ is analytic with continuous derivative $f'(z)$. C - (7)

Consider the complex function $f(x,y)$ of two real variable.

The complex variable $z = x+iy$ and its conjugate $\bar{z} = x-iy$

we can write

$$x = \frac{1}{2}(z+\bar{z})$$

$$y = \frac{1}{2i}(z-\bar{z})$$

we could obtain

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

polynomials: defn:

The sum and product of two analytic function are again analytic.

It follows that the every polynomials.

$$p(z) = a_0 + a_1 z + \dots + a_n z^n$$

is an analytic function. It is called a polynomial

i.e) p(z) [illegible]

where $p_1(z)$ is a polynomial of

P-(8) degree n-1.

$p(z) = a_n (z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_n)$

where the $\alpha_1, \dots \alpha_n$ are not necessarily distinct. C-(8)

From the factorization we conclu the p(z) not vanish. for any value of z.

difference from $\alpha_1, \alpha_2, \dots \alpha_n$

Theorem: 1 (Lucas's theorem).

If all zero of polynomial $p(z)$ lie in a form half plane. Then all the zeros of derivatives $p'(z)$ lie in the same half plane.

Proof:

Let all zero of polynomial $p(z)$ lie in the half plane.

To prove that

All the derivative of $p'(z)$ lie in the same half-plane.

$$\frac{p'(z)}{p(z)} = \frac{1}{z-\alpha_1} + \dots + \frac{1}{z-\alpha_n} \rightarrow (1).$$

Suppose that

the half plane H is defined as the part of the plane where

$\Rightarrow Im(z-a)/b < 0.$

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$\Rightarrow$ If α_k is in H and z is not

we have

$$\operatorname{Im}\frac{z-\alpha_k}{b} = \operatorname{Im}\frac{z-a}{b} - \operatorname{Im}\frac{\alpha_k - a}{b} > 0.$$

But the imaginary program reciprocal numbers have opposite sign.

C - (9)

under the same assumption

$\operatorname{Im} b(z-\alpha_k)^{-1} < 0$.

If this is true for all k, we conclude from eqn (8)

then i.e) $\operatorname{Im}\frac{bp'(z)}{p(z)} = \sum_{k=1}^{n} \operatorname{Im}\frac{b}{z-\alpha_k} < 0$

and a consequencely $p'(z) \neq 0$.

i.e) $p(z) \neq 0$

if the contains the zeros of $p(z)$ also contains the zeros of $p'(z)$.

Rational functions:

The return to case of rational functions

$$R(z) = \frac{p(z)}{Q(z)}$$

Given with the quotient of Q polynomials.

We assume and these is essential that $P(z)$ and $Q(z)$.

have know common factors and hence no common zeros.

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$R(z)$ will be given the value ∞ at the zeros of $Q(z)$.

(10)

poles:

C-(10)

~~The zeros of Q~~

The rational functions of the zeros of $Q(z)$ are called poles of $R(z)$ and the order of the pole is corresponding zeros are $Q(z)$.

The derivatives

$$R' = \frac{P'(z)\,Q(z) - Q'(z)\,P(z)}{Q(z)^2}$$

Rational function: defn:

A rational functions $R(z)$ of order p ~~p zeros~~ p has p zeros and p poles. and every eqn

$R(z) = a$.

has exactly p roots.

Linear transformation:

The rational function of order 1 is a linear fraction

$$S(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$$

with $\alpha\delta - \beta\gamma \neq 0$

This function it is called a linear transformation.

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Parallel translation:

(11) If the equation $w = S(z)$ has exactly one root. Such that $z = S^{-1}(w)$

$$= \frac{\alpha z + \beta}{\gamma z + \delta}$$ C-(11)

The transformation S and S^{-1} are the inverse to each other. The linear transformation $z+a$ is called a parallel translation.

Inversion:

The linear transformation $z+a$ is a parallel translation and the $\frac{1}{z}$ is called the inversion.

the fixed point 0 and ∞.

Interchanges 0 and ∞.

Singular part : (singular point):

Let the equation $R(z) = G(z) + H(z)$.

where $G(z)$ is a polynomial without constant term.

$H(z)$ is finite at ∞. The degree of is the order of the pole at the ∞ and the polynomial $G(z)$ is called the singular part of $R(z)$ at ∞.

Elementary theory of power series:

Sequences:

Contd Let the function $f : N \to R$

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convergent: (finite limit)

A sequence $\{a_n\}$ has the limit A if for every $\epsilon > 0$ there exists an n_0 such that $|a_n - A| < \epsilon \quad \forall\, n \geq n_0$

This sequence finite limit it is called convergent. (finite limit) C-(12)

divergent: (Infinite limit)

A sequence $\{a_n\}_{}^{\infty}$ has the limit A if for every $\epsilon > 0$. There exists an n_0 such that $|a_n - A| < \epsilon \quad \forall\, n \geq n_0$

This sequence infinite limit is called divergent.

cauchy sequence: (or) fundamental sequence:

A sequence $\{a_n\}_{}^{\infty}$ for any $\epsilon > 0$ there exist an n_0 such that $|a_n - a_m| < \epsilon$ when ever $n \geq n_0$ and $m \geq n_0$

Note:

A sequence is convergent iff it is a cauchy sequence.

Proof:

let $a_n \to A$

we can find n_0 such that

for $m, n \geq n_0$ if

P-13 it follows by the triangle inequality that $|a_n - a_m| \le |a_n - A| + |a_m - A| < \epsilon$

Limes superior: (limit superior) and limit inferior: C-13

A real sequence $\{\alpha_n\}_n$ we share sets $a_n = \max\{\alpha_1, \alpha_2, \ldots \alpha_n\}$

i.e) a_n is the greatest to the numbers $\{\alpha_1, \alpha_2 \ldots \alpha_n\}$.

The sequence $\{a_n\}_1^\infty$ is non decreasing it has a limit A, which is finite (or) equal to $+\infty$

Then the number A_1 is least upper bound (or) Supremum

~~Least upper bound~~

Limit inferior:

A real sequence $\{\alpha_n\}_n^\infty$ we share sets $a_n = \min\{\alpha_1, \alpha_2 \ldots \alpha_n\}$

i.e) a_n is the lowest to the numbers $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$.

The sequence $\{a_n\}_1^\infty$ is non-decreasing it has a limit A. which is infinite (or) equal to $-\infty$.

Then the number A_1 is greatest lower bound (or) infimum.

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(14) properties:

(i) $\underline{\lim}\, \alpha_n + \underline{\lim}\, \beta_n \leq \underline{\lim}\, (\alpha_n + \beta_n) \leq \underline{\lim}\, \alpha_n + \overline{\lim}\, \beta_n$

(ii) $\underline{\lim}\, \alpha_n + \overline{\lim}\, \beta_n \leq \overline{\lim}\, (\alpha_n + \beta_n) \leq \overline{\lim}\, \alpha_n + \overline{\lim}\, \beta_n$

proof:

Now, we written to the sufficiency of cauchy's condition.

C-14

Form $|\alpha_n - \alpha_{n_0}| < \varepsilon$

we obtain

~~$|\alpha_n| <$~~

$|\alpha_n| < |\alpha_{n_0}| + \varepsilon$ for $n \geq n_0$

and its follows that

$A = \overline{\lim}\, \alpha_n$ and $a = \underline{\lim}\, \alpha_n$ are both finite.

If $a = \frac{(A-a)}{3}$

and determine the corresponding to n_0.

By defn of a and A there exists an $\alpha_n < a + \varepsilon$ and

$\alpha_m > A - \varepsilon$ with $m, n \geq n_0$.

It follows that

$A - a = (A - \alpha_m) + (\alpha_m - \alpha_n) + (\alpha_n - a) < 3\varepsilon$

Hence the sequence is consistency

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P-15

Therefore

and

(i) $\underline{\lim}\, \alpha_n + \underline{\lim}\, \beta_n \leq \underline{\lim}\, (\alpha_n + \beta_n) \leq \underline{\lim}\, \alpha_n +$ $\overline{\lim}\, \beta_n$

$A = \overline{\lim}\, a_n$ and $a = \underline{\lim}\, \alpha_n$

$\cancel{A + \underline{\lim}\, \beta_n}$

C-15

$a + a \leq \underline{\lim}\, (\alpha_n + \beta_n) \leq a + A$

$2a \leq \underline{\lim}\, (\alpha_n + \beta_n) \leq a + A$

$a \leq \dfrac{\underline{\lim}\, (\alpha_n + \beta_n)}{2} \leq \dfrac{a + A}{2}$

(ii) $\underline{\lim}\, \alpha_n + \overline{\lim}\, \beta_n \leq \overline{\lim}\, (\alpha_n + \beta_n)$

$\leq \overline{\lim}\, \alpha_n + \overline{\lim}\, \beta_n$

$a + A \leq \overline{\lim}\, (\alpha_n + \beta_n) \leq A + A$

$a + A \leq \overline{\lim}\, (\alpha_n + \beta_n) \leq 2A.$

Series:

Infinite series:

An infinite series is a formal infinite sum

$a_1 + a_2 + \dots + a_n + \dots$

associated with the series

$a_1 + a_2 + \dots + a_n$

~~Uniform convergent:~~

Absolutely convergent:

Let the series of A defined on a sur

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$\leq |a_n| = |a_n|$

i.e) $|a_n + a_{n+1} + \dots + a_{n+b}|$

$\leq |a_n| + |a_{n+1}| + \dots + |a_{n+b}|$

of the terms convergence is said to be a absolutely convergence.

%. 2m **Uniform convergent:**

A sequence of $\{f_n(x)\}$ convergence uniformly to $f(x)$ on the set ~~a~~ E.

if two every $\varepsilon > 0$ there exists an n_0 such that

$|f_n(x) - f(x)| < \varepsilon \quad \forall\, n \geq n_0$.

and all x in E.

Ex:

The limit functions of a uniformly convergence of continuous functions is itself continuous.

Proof:

Suppose that

the function $f_n(x)$ of continuous such that uniformly to $f(x)$ on the set E for any $\varepsilon > 0$.

we are find and n such that

$|f_n(x) - f(x)$ [illegible]

Let x_0 be [illegible]

Because $f_n(x)$ continuous

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we can find $\delta > 0$

Such that

$$|f_n(x) - f_n(x_0)| < \epsilon/3 \quad \forall x \in E$$

with

$$|x - x_0| < \delta$$

under the ~~same~~ condition on x

it follows that

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3$$

$$< \epsilon$$

There $f(x)$ is continuous at x_0.

Continuous function (or) continuous:

The sequence $\{f_n(x)\}$ converges uniformly on E.

iff and only if every $\epsilon > 0$ there exists an n_0..

Such that

$$|f_m(x) - f_n(x)| \leq \epsilon \quad \forall m, n \geq n_0$$

and the all x in E.

power series:

A power series is of the form

$$a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n + \dots$$

where ~~[illegible]~~ and the variable of z of complex.

The [illegible] more generally

P-(18)

we may consider series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

which are power series with respect the centre z.

Geomentric series:

We consider the geomentric series

$$1+z+z^2+\dots+z^n+\dots$$

whose partial sums can be wr in the form

$$1+z+\dots+z^{n-1} = \frac{1-z^n}{1-z}$$

Since

$z^n \to 0$ for $|z|<1$ and $|z^n|$ for $|z| \geq 1$

Then the geomentric series converges $\frac{1}{(1-z)}$ for $|z|<1$.

Divergence for $|z| \geq 1$

Radius of convergent:

If every power series $a_0+a_1z \dots a_nz$

There exists a number R, $0 \leq R \leq \infty$

It is called the Radius of convergent

Circle of convergent:

A power series is of the form

$$a_0+a_1z+a_2z^2+\dots+a_nz^n,$$

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of the circle $|z| = R$ is called the circle of convergence.

Theorem:

For every power series is of the form

$a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n + \dots \longrightarrow (1)$

There exists a number R, $0 \le R \le \infty$ the radius of convergent with the following properties (i) the series convergent absolutely for every z with $|z| < R$ if $0 \le \rho < R$ the convergent is of uniform for $|z| \le \rho$

(ii) If $|z| > R$ the terms of the series are unbounded and the series is consequently divergent.

(iii) In $|z| < R$ the sum of the series is an analytic function the derivative can be obtained by termwise different and the divided series has the same values of convergent.

Proof:

The circle $|z| = R$ is called the circle of convergent we shall shows that the assertion in the theorem are true if R is chosen according to the formula.

$$\frac{1}{R} = \lim_{n \to \infty} \sup \sqrt[n]{|a_n|}$$

If $|z| < R$ we can find ρ, so that $|z| < \rho < R$

Then $1/\rho > 1/R$ and by the defn of lim

Superior.

There exists an n_0 such that

$|a_n|^{1/n} < 1/\rho$.

$|a_n| < 1/\rho^n$ for $n \geq n_0$.

It follows that

$|a_n z^n| < (|z|/\rho)^n$ for range n.

To prove the uniform convergence for

$|z| \leq \rho < R$.

we choose a ρ' with $\rho < \rho' < R$.

and find $|a_n z^n| \leq (\rho/\rho')$ for $n \geq n_0$

Since the informed majorant is convergent and has constant term we conclude by weierstrass M test that the power series is uniformly convergent.

If $|z| < R$

we choose ρ so that $R < \rho < |z|$.

Since $1/\rho < 1/R$ there are arbitrarily large n such that

$|a_n|^{1/n} > 1/\rho$

$|a_n| > 1/\rho^n$

Thus $|a_n z^n| > (|z|/R)^n$ for infinitely many n, and the taken are unbounded

The derived series

$\sum_1^\infty n a_n z^{n-1}$ has the same radius of convergence because

$\sqrt[n]{n} \to 1$

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Proof:

Set $\sqrt[n]{n} = 1 + \delta_n$.

Then $\delta_n \geq 0$

$n = (1+\delta_n)^n \geq 1 + \frac{1}{2} n(n-1)\delta^2_n$.

This gives $\delta^2_n < 2/n$

and hence $\delta_n \to 0$

for $|z| < \rho$

we shall write

$f(z) = \sum_0^\infty a_n z^n$

$= s_n(z) + R_n(z)$ ... (3)

where

$s_n(z) = a_0 + a_1 z + \dots + a_{n-1} z^{n-1}$

$R_n(z) = \sum_{k=n}^{\infty} a_k z^k$

and also

$f_1(z) = \sum_1^\infty n a_n z^{n-1}$

$= \lim_{n\to\infty} s'_n(z)$

we have to show that $f'(z) = f_1(z)$.

$$\frac{f(z)-f(z_0)}{z-z_0} - f_1(z_0) = \left(\frac{s_n(z)-s_n(z_0)}{z-z_0} - s_n'(z_0)\right)$$

$$+ (s_n'(z_0) - f_1(z_0)) + \left(\frac{R_n(z)-R_n(z_0)}{z-z_0}\right)$$

assume that $z \neq z_0$ and $|z|$

$|z_0| < \rho < R$

$\sum_{k=n}^{\infty} a_k (z^{k-1} + z^{k-2} z_0 + \dots + z_0^{k-2} + z_0^{k-1}$

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and we conclude that

$$\left|\frac{R_n(z) - R_n(z_0)}{z - z_0}\right| \leq \sum_{k=n}^{\infty} k|a_k|\rho^{k-1}$$

The expression on the right is the remainder term in a convergent series

Hence we can find n_0 such that

$$\left|\frac{R_n(z) - R_n(z_0)}{z - z_0}\right| < \varepsilon/3 \longrightarrow (5)$$

for $n \geq n_0$

there is also an n_1 such that

$|S_n'(z_0) - f_1(z_0)| < \varepsilon/3$ for $n \geq n_1$.

choose a fixed $n \geq n_0, n_1$.

By the defn of derivative

we can find $\delta > 0$ such that

$0 < |z - z_0| < \delta$.

$$\Rightarrow \left|\frac{S_n(z) - S_n(z_0)}{z - z_0}\right| < \varepsilon/3. \longrightarrow (6)$$

(5) and (6)

$$\Rightarrow \left|\frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0)\right| < \varepsilon.$$

where $0 < |z - z_0| < \delta$.

we have proved that $f'(z_0)$ exists and equals $f_1(z_0)$

Since the reasoning can be repeated.

we have in reality prove much more.

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A power series with positive radius of convergence has derivatives of all order and they are given explicitly by

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

$$f'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \cdots$$

$$f''(z) = 2a_2 + 6a_3 z + 12a_4 z^2 + \cdots$$

$$\vdots$$

$$f^{(k)}(z) = k!\,a_k + \frac{(k+1)!}{1!} a_{k+1} z + \frac{(k+2)!}{2!} a_{k+2} z^2 + \cdots$$

In particular if $z=0$ look at the last line we see that

$$a_k = f^{(k)}(0)/k!$$

and the power series

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!} z^2 + \cdots + \frac{f^{(n)}}{n!} z^n + \cdots$$

This is the familiar Taylor-maclaurin development but we have proved it only under the ~~association~~ assumption that $f(z)$ has a power series development. we know that the development is uniquely determined if it exist but the main part is still missing, namely, that every analytic function has a Taylor development

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Abel's limit theorem:

Statement:

If $\sum_0^\infty a_n$ converges, then $f(z) = \sum_0^\infty a_n z^n$ tends to $f(1)$ as z approaches 1 in such way that $|1-z|/(1-|z|)$ remains bounded.

proof:

We may assume that $\sum_0^\infty a_n = 0$.

for this can be ~~attaind~~ attained by adding a constant to a_0.

We write $S_n = a_0 + a_1 + \dots + a_n$

and make use of identity (summation by parts).

$S_n(z) = a_0 + a_1 z + \dots + a_n z^n$

$= S_0 + (S_1 - S_0) z + \dots + (S_n - S_{n-1}) z^n.$

$= S_0(1-z) + S_1(z-z^2) + \dots + S_{n-1}(z^{n-1} - z^n) + S_n z^n.$

$= (1-z)(S_0 + S_1 z + \dots + S_{n-1} z^{n-1}) + S_n z^n$

But $S_n z^n \to 0$. So we obtain the representation

$$\boxed{f(z) = (1-z) \sum_0^\infty S_n z^n}$$

we are assuming that

$|1-z| \le k(1-|z|)$, say and that $S_n \to 0$.

Choose m so large that $|S_n| < \varepsilon$ for $n \ge m$.

The remainder of the series $\sum S_n z^n$, from $n = m$ on.

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is then dominated by the geometric series.

$$\varepsilon \sum_{m}^{\infty} |z|^n = \varepsilon |z|^m / (1-|z|) < \varepsilon/(1-|z|)$$

It follows that

$$|f(z)| \leq |1-z| \left| \sum_{0}^{m-1} s_k z^k \right| + k\varepsilon.$$

The first term on the right can be made arbitarily small by choosing z sufficiently close to 1 and

we conclude that

$f(z) \to 0$ when $z \to 1$

Subject to the stated restriction.

CONFORMALITY:

1: closed and arc curve:

The equation of an arc γ in the plane is most conveniently given in a parametric form

$x = x(t),\ y = y(t)$

where t runs through an interval

$\alpha \leq t \leq \beta$.

and $x(t)$ and $y(t)$ are continuous function.

The complex notation $z = z(t) = x(t) + iy(t)$.

The mapping it is denoted by $z = f(t)$.

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Defn of Differentiable (or) regular closed curve under (or) of Simple closed curve (or) Jordan curve:

Let the eqn is

$z = z(t), \ \alpha \le t \le \beta.$

to choose a point t_0 from the interval (α, β) and defined a new closed curve whose the eqn is $z = z(t)$.

for $t_0 \leqq t \leqq \beta$ and

$z = z(t - \beta + \alpha)$ for $\beta \leqq t \le t_0 + \beta - \alpha$.

is equation it is called differentiable curve.

Jordan arc is simple:

let $z(t_1) = z(t_2)$

only for $t_1 = t_2$ and arc is a closed curve if the end points co-incide $z(\alpha) = z(\beta)$

It is called a Jordan arc (or) Simple arc.

opposite arc :

The opposite arc of

$z = z(t), \ \alpha \leqq t \leqq \beta$ is the arc $z = z(-t), \ -\beta \leqq t \le -\alpha.$

It is called a opposite arc.

point curve:

opposite arcs are compined denoted by γ and $-\gamma$.

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Sometimes by r and r

depending on the connection of the constant function $z(t)$ it is called the point curve.

Circle:

A circle C originally defined as locus $|z-a| = r$

can be considered as a closed curve with the eqn

$$z = a + re^{it}, \quad 0 \le t \le 2\pi$$

is called a circle.

This eqn is convenient between C and

Analytic functions in Regions:

The complex value function $f(z)$ defined on an set Ω (omega) is said to be analytic in Ω.

If it has a derivative at each point of Ω

Thm:

An analytic function in a region Ω whose derivative vanish identically must reduce to a constant. The same is true if either the real part, imaginary part, modulus (or) the argument is constant.

proof:

The vanish of the derivatives

$\Rightarrow \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y};$

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are all zero.

It is follows that u and v are constant on any line segment in Ω. which is parallel to one of the coordinal co-ordinate axes.

we conclude that $u+iv$ is constant.

If u or (or) v is constant.

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.$$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

and hence $f(z)$ must be constant.

If u^2+v^2 is a constant.

we obtain

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0.$$

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = -u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0.$$

this equation permit the conclusion

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0.$$

unless the termitnant u^2+v^2 vanishes.

But, if $u^2+v^2=0$ at a single point it is constantly zero and $f(z)$ vanishes identically.

Hence $f(z)$ is, an, any case a

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constant

Finally

If $\arg(f(z))$ is constant

$\Rightarrow u = kv$, with constant k.

unless v is identically zero.

But $u+kv$ is the real part of $(1+ik)f$

we conclude the again that

f must reduce to a constant.

Therefore $f(z)$ is constant on each component of Ω.

Conformal mapping:

Suppose that an arc γ with the eqn $z = z(t)$, $\alpha \le t \le \beta$ is contained in a Jordan region Ω and let $f(z)$ be defined and continuous in Ω. Then the equation

$\omega = \omega(t) = f(z(t))$.

defined on arc γ in the w-plane which may be called the image of γ.

Consider the case of an $f(z)$ which is analytic in Ω;

If $z'(t)$ exists

$\therefore \omega'(t) = f'(z(t))\, z'(t) \longrightarrow (1)$.

This eqn of a point z_0

at $z(t_0)$ with $z'(t_0) \neq 0$ and

$f'(z_0) \neq 0$.

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The first conclusion is the $w'(t_0)$ to hence γ' has a tangent at $w_0 = f(z_0)$.

$\therefore \arg(w'(t_0)) = \arg(f'(z_0)) + \arg(z'(t_0))$ → ②

If then the mapping by $w = f(z)$ is said to conformal at all points with $f'(z) \neq 0$.

Indirectly conformal:

If the eqn

This case $\overline{f(z)}$ is analytic a mapping by the conjugate of non analytic function with a non-vanishing derivative is said to be indirectly conformal.

Length and Area:

The conformal mapping $f(z)$ the length of an infinite line of segment at the point z. we conclude the length of a differentiable arc γ with the equation

$z = z(t) = x(t) + iy(t)$, $a \leq t \leq b$ is given

$$L(\gamma) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$= \int_a^b |z'(t)|\, dt$$

The image curve γ' is determined by $w = w(t) = f(z(t))$. with the derivative

$w'(t) = f'(z(t)) z'(t)$

It's length is this $L(\gamma') = \int_a^b |f'(z(t))|\, |z'(t)|\, dt$.

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$$\left.\begin{aligned} l(\gamma) &= \int_{\gamma} |dz| \\ l(\gamma') &= \int_{\gamma} |f'(z)|\,|dz| \end{aligned}\right\} \rightarrow (1).$$

~~A(E) - f~~

now, that E be a point set in the plane whose area.

$$A(E) = \iint_E dx\,dy$$

It can be evaluate as a double Riemann integral

If $f(z) = u(x,y) + iv(x,y)$ is a bijective differentiable mapping.

The rule for changing Integration variables:

The area of the image $E' = f(E)$ is given by

$$A(E') = \iint |u_x v_y - u_y v_x|\,dx\,dy$$

But, If $f(z)$ is a conformal mapping and open set containing in E.

Then $u_x v_y - u_y v_x = |f'(z)|^2$

by the Cauchy's Riemann equation. we obtain

$$A(E') = \iint_E |f'(z)|^2\,dx\,dy \rightarrow (2)$$

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The formulas for eqn. (1) and (2) are complex analysis for the geometric functions.

Linear transformation:

A linear transformation

rational $w = S(z) = \frac{az+b}{cz+d}$ with $ad-bc \neq 0$. → (1).

has an inverse

$z = S^{-1}(w) = \frac{dw-b}{-cw+a}$.

the special values

$S(\infty) = a/c$ and $S(-d/c) = \infty$

as the limit for $z \to \infty$ and $z \to -d/c$ from the eqn (1) normalized if $ad-bc=1$.

The linear transformation is the use for homogeneous coordinates

$z = z_1/z_2$, $w = w_1/w_2$

$w = S_2$ if $w_1 = az_1 + bz_2$

$w_2 = cz_1 + dz_2$

in matrix notation

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

The composite transformation

$w = S_1 S_2 z$

be use of the matrices corresponded to S_1, S_2.

It is immediate that $S_1 S_2 \in$ the matrix product.

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$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

$$= \begin{pmatrix} a_1a_2+b_1c_2 & a_1b_2+b_1d_2 \\ c_1a_2+d_1c_2 & c_1b_2+d_1d_2 \end{pmatrix}$$

All a linear transformation from a group from a associative law for a arbitary transformation.

The identity $w=z$. is a linear transformation

parallel translation:

The simples linear transformation belong to matrices of the form.

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The first of this $w=z+\alpha$. is called a parallel translation.

homothetic transformation:

The simples linear transform belong to the matrices of the form

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The second transformations $w=kz$ is a rotation if $|k|=1$.

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and it is called a homothetic transformation

Inversion:

If $w = 1/z$ is called an inversion if $c \neq 0$

we can write

$$\frac{az+b}{cz+d} = \frac{bc-ad}{c^2(z+d/c)} + a/c.$$

This eqn is called a inversion.

Cross ratio:

Given the three distict points z_1, z_2, z_3, z_4 in the extended plane, there exists a linear transformation which carries them into $1, 0, \dots \infty$ In this order.

If none of the points is ∞, S will be given by

$$S_z = \frac{z-z_3}{z-z_4} : \frac{z_2-z_3}{z_2-z_4} \quad S, S_1$$

If z_2, z_3 or $z_4 = \infty$

The transformation reduces to

$$\frac{z-z_3}{z-z_4}, \quad \frac{z_2-z_4}{z-z_4}, \quad \frac{z-z_3}{z_2-z_3}$$

T where another linear transfo with the same property

ST^{-1} invariant:

The transformation it is true for identity transformation at $S=T$

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we conclude that S is uniquely determined.

Cross ratio:

The cross ratio (z_1, z_2, z_3, z_4) is the image of z_1 under the linear transformation which carries z_2, z_3, z_4, in to $(1, 0, \infty)$.

Theorem: 12

If z_1, z_2, z_3, z_4 are distinct points in the existence plane and T any linear transformation then $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$

proof:

Let (z_1, z_2, z_3, z_4) are distinct points in the existence plane and T is any linear transformation,

To prove that

$(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$.

If $Sz = (z_1, z_2, z_3, z_4)$,

then ST^{-1} carries Tz_2, Tz_3, Tz_4 into $1, 0, \infty$.

by using the defn

$(Tz_1, Tz_2, Tz_3, Tz_4) = ST^{-1}(Tz_1)$

$= Sz_1$

$= (z_1, z_2, z_3, z_4)$

Therefore $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$

P-36 Theorem: 13

The cross ratio (z_1, z_2, z_3, z_4) is real iff the four points lie on a circle or lie on a straight line.

Proof:

Part (i)

Assume that

The cross ratio (z_1, z_2, z_3, z_4) is real.

To prove that:

The four points lie on a circle or lie on a straight line.

By using elementary geometry function

we obtain

$$\arg(z_1, z_2, z_3, z_4) = \arg\frac{(z_1 - z_3)}{(z_1 - z_4)} - \arg\left(\frac{z_2 - z_3}{z_2 - z_4}\right).$$

And the points lie on the circle the difference of angles is either zero or $\pm\pi$.

Assume that:

The four points lie on a circle or straight line.

To prove that

The cross ratio (z_1, z_2, z_3, z_4) and real.

Let $Tz = (z, z_2, z_3, z_4)$ is real on the image of the real axis under the transformation.

T^{-1} the values of $w = T^{-1}z$ for real z satisfy the eqn.

$$Tw = \overline{Tw}$$

This condition is of the form

$$\frac{aw+b}{cw+d} = \frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}}$$

we obtain

$$(\bar{a}c - c\bar{a})|w^2| + (a\bar{d} - c\bar{b})w + (b\bar{c} - d\bar{a})\bar{w} + b\bar{d} - d\bar{b} = 0.$$

If $a\bar{c} - c\bar{a} = 0$ is the equation is straight line

If $a\bar{c} - c\bar{a} \neq 0$ we obtain

eqn $\left| w + \frac{\bar{a}d - \bar{c}b}{\bar{a}c - \bar{c}a} \right| = \left| \frac{cd - bc}{\bar{a}c - \bar{c}a} \right|$

which is eqn is circle.

Symmetry: (defn for symmetry and theorem)

The points z and z^* are said to be symmetry with respect to the circle c through z_1, z_2, z_3 iff $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$.

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Proof :

part - I :

assume that :

The points z and z^* are said to be symmetry with respect to the circle c. through (z_1, z_2, z_3).

To prove that

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$$

The mapping which has is z into z^* is a (1-1) correspondence it is called a reflection w.r.t. respect to c $(z \sim z^*)$

C is straight line.

Then we can choose

$z_3 = \infty$. and the symmetric becomes

$$\frac{z^* - z_2}{z_1 - z_2} = \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2} \rightarrow (1)$$

Taking absolutely values we obtain

$|z^* - z_2| = |z - z_2|$.

Here z_2 can be any finite point we conclude that

z and z^* are equal distance from the points on c.

By using eqn (1)

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$$\operatorname{Im}\frac{z^* - z_2}{z_1 - z_2} = -\operatorname{Im}\frac{z - z_2}{z_1 - z_2} \longrightarrow (2)$$

from eqn ① and ②.

Therefore

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$$

Conversely,

assume that:

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$$

To prove that

The points z and z^* are said to be symmetric with respect to the circle c. through (z_1, z_2, z_3)

We consider now the case of a finite circle c of centre a and radius R,

Symmetric use of the invariance of the cross ratio ... conclude as follows.

$$\overline{(z, z_1, z_2, z_3)} = \overline{(z-a, z_1-a, z_2-a, z_3-a)}$$

$$= \left(\bar{z}-\bar{a}, \frac{R^2}{z_1-a}, \frac{R^2}{z_2-a}, \frac{R^2}{z_3-a}\right)$$

$$= \left(\frac{R^2}{\bar{z}-\bar{a}}, z_1-a, z_2-a, z_3-a\right)$$

$$= \left(\frac{R^2}{\bar{z}-\bar{a}} + a, z_1, z_2, z_3\right)$$

This equation show that the symm

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point of

z is $z^* = \dfrac{R^2}{(\bar{z}-\bar{a})} + a$ (or) that z and z^*

satisfy the relation

(25) $\Rightarrow (z^*-a)(\bar{z}-\bar{a}) = R^2$

The product of $|z^*-a|, |z-a|$ of the distances to the centre a is hence R^2 (its positive)

Further the ratio $(z^*-a)/(z-a)$ is positive, which means that z and z^* are situated on the same half line from a.

There is a simple geometric construction for the symmetric point of z. we note that the symmetric point of a is ∞.

Thm:

The symmetry principle.

Statement:

If a linear transformation carries a circle c_1 into a circle c_2, then it transforms any pair of symmetric points with respect to c_1 into a pair of symmetric points with respect to c_2.

Proof:

If c_1 or c_2 is the real axis. The principle follows from [illegible]

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of symmetry.

The point z and z^* are said to be symmetric with respect to the circle c through z_1, z_2, z_3

if $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$

$\Rightarrow$ The principle of symmetry in put to practical use in the problem of finding the linear transformation.

$\Rightarrow$ which carry a circle c into a circle c'.

$\Rightarrow$ we can always determine the transformation by requiring that three points z_1, z_2, z_3, z_4 on c go over into the three points w_1, w_2, w_3 on c'.

$\Rightarrow$ The transformation is then

$$(w_1, w_2, w_3) = (z_1, z_2, z_3).$$

$\Rightarrow$ But the transformation is also determined if we prescribe that a point z_1 on c shall correspond a point w_1 on c'.

$\Rightarrow$ we know that z_2^* (The symmetric point of z_2 w.r.to c) correspond to w_2^* (The symmetric point of w_2 w.r.to to c')

hence the transformation will be obtained from the relation (w, w_1, w_2, w_2^*)

$$= (z, z_1, z_2, z_2^*).$$

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Oriented circles: defn:

Because $S(z)$ is analytic and

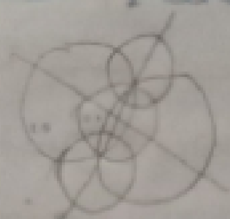
$$S'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$$ the mapping

$w = S(z)$ is conformal for $z \neq -d/c$ and

families of circles: defn:

consider a linear transformation of the form

$$w = k \cdot \frac{z-a}{z-b}$$



Here $z = a$ corresponds to $w = 0$ and $z = b$ to $w = \infty$.

It follows that the straight line through the origin of the w-plane are images of the circles through a and b on the other hand, the concentric circles about the origin,

$|w| = \rho$ correspond to circles with the equation

In the plane with the [illegible] (i) There is exactly one C_1 and C_2 through each [illegible] the limit point (ii) every C_1 meets every C_2 at the right angle (iii) reflection in C_1 transforms every C_2 into itself and every [illegible] into another (iv) The limit points are symmetric w.r.t. the [illegible] [illegible] to any other circle

$$\left| \frac{z-a}{z-b} \right| = \rho/|k|.$$

These are the circles of Apollonius with limit points a and b.

Defn: Loxodromic:

The circles C_1 and C_2 are carried into the circles C_1' and C_2' determined by a' and w'.

we suppose now that $a = a'$ is the only fixed point. Then $w = w'$

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and we can write

$$\frac{w}{w-a} = \frac{w}{z-a} + c.$$

By this transformation the configuration consisting of the circles c_1 and c_2 is mapped upon itself.

A linear transformation that is neither hyperbolic, elliptic, nor parabolic is said to be loxodromic.

UNIT: II

complex integration:

Fundamental theorems:

Line integral:

The most immediate generalization of a real integral to the definite integral of a complex function over a real interval.

If $f(t) = u(t) + iv(t)$

define in an interval (a,b) be said

by defn $\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt \rightarrow (1)$

This an interval has most of properties of the real integral.

In particular if $c = \alpha + i\beta$ is a complex constant is we obtain

$$\int_a^b c\, f(t)\,dt = c\int_a^b f(t)\,dt \rightarrow (2)$$

for fourth members are equal to

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$\int_a^b (\alpha u - \beta v)\,dt + i(\alpha v + \beta u)\,dt$

The $a \leq b$.

The fundamental ~~equat~~ inequality

$$\left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|\,dt \rightarrow (3).$$

holes for arbitrary constants $f(t)$ to see this we choose $c = e^{-i\theta}$ with the real part of θ in eqn (2)

we get

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integration of complex functions
$$Re\left[e^{-i\theta}\int_a^b f(t)\,dt\right] = \int_a^b Re\left[e^{-i\theta} f(t)\right]dt$$
$$\leq \int_a^b |f(t)\,dt.$$

For $\theta = \arg \int_a^b f(t)$.

for the absolutely values for eqn (3)

complex integration defn.

$\Rightarrow$ The differentiable arc γ with the equation $z = z(t)$, $a \leq t \leq b$.

$\Rightarrow$ The function $f(z)$ is defined an continuous on γ.

Then $f(z(t))$ also continuous.

$$\therefore \int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt \rightarrow (4)$$

This is our defn of the complex line integral of $f(z)$ extended over the arc γ.

In the right hand member
if $z'(t)$ is not continuous ...

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the interval of integration has to be substituted in the obvious manner.

⇒ whenever a line integral over an arc γ is considered, let it be tacitly understood that γ is piecewise differentiable

⇒ The most important property of the integral ④ is its invariance under a change of parameter

⇒ A change of parameter is determined by an increasing function $t = t(\tau)$.

⇒ which maps an interval $\alpha \le \tau \le \beta$.

onto $a \le t \le b$.

⇒ we assume that $t(\tau)$ is piecewise differentiable by the [illegible] rule.

① is not defined if $\int_a^b f\,dt = 0$.
but then there is nothing to prove

⇒ For changing the variable of integration

$$\int_a^b f(z(t))\,z'(t)\,dt = \int_\alpha^\beta f(z(t(\tau)))\,z'(t(\tau))\,t'(\tau).$$

⇒ But $z'(t(\tau))t'(\tau)$ is the derivative of $(t(\tau))$ with respect to τ, and hence the integral ④ has the same value whether γ be represented by the equation

$z = z(t)$ (or) by the equation

$z = z(t(\tau))$.

we defined the opposite arc $-\gamma$ by the eqn $z = z(-t)$, $-b \le t \le -a$

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we have thus

$$\int_{-\gamma} f(z)dz = \int_{-b}^{-a} f(z(-t)) - z'(-t)) dt$$

$$\int_{0}^{a} f(z(t)) \, z'(t) \, dt.$$

we concludes that

$$\int_{-\gamma} f(z)dz = -\int_{\gamma} f(z)dz \longrightarrow (5).$$

A subdivision can be indicated by a symbolic eqn

$$\gamma = \gamma_1 + \gamma_2 + \dots + \gamma_n$$

corresponding integrals satisfy the Relation

$$\int_{\gamma_1+\gamma_2+\dots+\gamma_n} f dz = \int_{\gamma_1} f dz + \int_{\gamma_2} f dz + \dots + \int_{\gamma_n} f dz \; \longrightarrow (6).$$

finally,

the integral the most contained defn

Is by double conjunction

$$\int_{\gamma} f d\bar{z} = \overline{\int_{\gamma} \bar{f} dz}$$

line integral is respect to x or y can be return as

$$\int_{\gamma} f dx = \frac{1}{2}\left(\int_{\gamma} f dz + \int_{\gamma} f d\bar{z}\right)$$

$$\int_{\gamma} f dy = \frac{1}{2i}\left(\int_{\gamma} f dz - \int_{\gamma} f d\bar{z}\right)$$

with $f = u + iv$ by using the integral

eqn (4) can be written in the

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$\int_\gamma (u\,dx - v\,dy) + i\int_\gamma (u\,dy + v\,dx) \longrightarrow ①$

which seperate real and imaginary part

we have started by the defn of integrals of the form

$\int_\gamma p\,dx + q\,dy$

by eqn ① of the essential different line integral by integration with respect to the arc length.

$$\int_\gamma f\,ds = \int_\gamma f\,|dz| = \int_\gamma f(z(t))\,|z'(t)|\,dt \quad \rightarrow ⑧$$

This is the integral is again independent of the parameter by using eqn ⑤.

$$\int_{-\gamma} f\,|dz| = \int_\gamma f\,|dz|.$$

by eqn ⑥ of the inequality

$$\left|\int_\gamma f\,dz\right| \leq \int_\gamma |f|\,.\,|dz| \rightarrow ⑦.$$

is a consequence of ⑤.

for $f=1$ sub the integral of eqn ⑧.

reduces to

$\int_\gamma |dz|$ by using the defn of length of

by using the example.

Ex: 1

The parametric of the eqn

$z(t) = a + \rho e^{it}, \quad 0 \leq t \leq 2\pi.$

$z'(t) = i\rho e^{it}.$

Proof:

Let the parametric eqn

$z = z(t) = a + \rho e^{it}$

Diff w.r.to t

$z'(t) = i\rho e^{it}$

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48 $\int_0^{2\pi} |z'(t)|\,dt = \int_0^{2\pi} \rho\,dt = 2\pi\rho.$

Rectifiable arcs:

The length of an arc can also be defined as the least upper bound of all sums

$$|z(t_1) - z(t_0)| + |z(t_2) - z(t_1)| + \dots + |z(t_n) - z(t_{n-1})| \quad \text{①}$$

where $a = t_0 < t_1 < \dots < t_n = b$.

If the least upper bound is infinite.

we say that the arc is rectifiable.

~~The differenti~~ The differential of an arc of rectifiable.

because.

$$|x(t_k) - x(t_{k-1})| \leq |z(t_k) - z(t_{k-1})|,$$

$$|y(t_k) - y(t_{k-1})| \leq |z(t_k) - z(t_{k-1})|$$

$$|z(t_k) - z(t_{k-1})| \leq |x(t_k) - x(t_{k-1})| + |y(t_k) - y(t_{k-1})|$$

by using the eqn ① is

$$|x(t_1) - x(t_0)| + \dots + |x(t_n) - x(t_{n-1})|$$

$$|y(t_1) - y(t_0)| + \dots + |y(t_n) - y(t_{n-1})|$$

are bounded at the same time.

We say that the functions $x(t)$, $y(t)$ are of bounded variation.

Bounded variation:

An arc $z = z(t)$ is rectifiable iff the real and imaginary part $z(t)$, are of bounded variation

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(49) If γ is rectifiable and $f(z)$ is continuous on γ.

by using the eqn

$$\int_\gamma f\,ds = \int_\gamma f|dz| = \int_\gamma f(z(t))\,|z'(t)|\,dt. \rightarrow ⑤$$

This eqns of limit of integrals of the sum

$$\boxed{\int f\,ds = \lim \sum_{k=1}^{n} f(z(t_k))\,|z(t_k) - z(t_{k-1})|}$$

definite integral. defn

This is the equation of ⑤ it is called definite integral.

Line integral of the function of arc:

The general integral of line of the form

$$\int_\gamma p\,dx + q\,dy \rightarrow ①$$

In otherwords γ_1, γ_2 have as a same initial point, and the same end point

we obtained the eqn of the form

$$\int_{\gamma_1} p\,dx + q\,dy = \int_{\gamma_2} p\,dx + q\,dy \rightarrow ②$$

$$\gamma = \gamma_1 = \gamma_2$$

if γ is a closed curve

Then γ and $-\gamma$ have the same end point and the integral depends only on the end point we obtained the eqn of the form

$$\int_\gamma = \int_{-\gamma} = -\int_\gamma \rightarrow ③.$$

we by using the eqn ③ $\int_\gamma = 0$

conversely

If γ_1 and γ_2 have the same end points then $\gamma_1 - \gamma_2$ is a closed curve.

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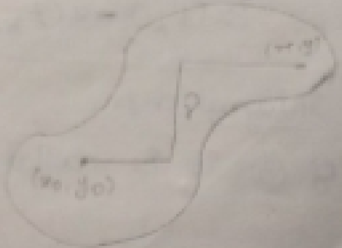
And if the integral over the curve
50 vanishes.

~~over vanishes~~ its follows that the eqn of the form

$$\int_{\gamma_1} = \int_{\gamma_2} \rightarrow (1)$$

Draw the diagram:



Thm: 1

The line integral $\int_\gamma P\,dx + q\,dy$, defined in Ω depends only on the end points of γ if and only if there exists a function $U(x,y)$ in Ω with the partial derivative $\partial U/\partial x = P$, $\frac{\partial U}{\partial y} = Q$.

Proof:

The sufficiency follows at once for if the condition is fulfilled we can write, with the usual notations,

$$\int_\gamma P\,dx + q\,dy = \int_a^b \left(\frac{\partial U}{\partial x} x'(t) + \frac{\partial U}{\partial y} y'(t)\right)dt$$

$$= \int_a^b \frac{d}{dt} U(x(t), y(t))\,dt$$

$$= U(x(b), y(b)) - U(x(a), y(a))$$

and the value of this difference depends only on the end points.

To prove that

Necessity we choose a fixed point $(x_0, y_0) \in \Omega$ join it to (x, y) by a polygon γ contained in Ω.

whose sides are parallel to the co-ordinate axes (Fig 4.1) and define a function by

$$U(x,y) = \int_\gamma P\,dx + q\,dy .$$

[Since the integral depends only on the end points the function is well defined.

=> Moreover if we choose the last segment of γ horizontal.

we can keep y constant and let x vary without changing the other segments]

=> on the last segment we can choose x for parameter and obtain.

$$U(x,y) = \int^x p(x,y)\,dx + \text{constant}.$$

The lower limit of the integral being irrelevant.

=> from this expression it follows at once that $\frac{\partial U}{\partial x} = p$.

In the same, by choosing the last segment vertical, we can show that

$$\frac{\partial U}{\partial y} = q$$

It is customary to write

$$dU = \left(\frac{\partial U}{\partial x}\right)dx + \left(\frac{\partial U}{\partial y}\right)dy$$ and to say that an expression $p\,dx + q\,dy$

which can be written in this form is an exact differential.

=> [Thus an integral depends only on the end points if and only if an integral is an exact differential].

52) $\Rightarrow$ observe that p, q and v can be either real or complex.

The function v, if it exists is uniquely determined up to an additive constant

by following the defn there must exists a function $F(z)$ in Ω with the partial derivatives

$$\frac{\partial F(z)}{\partial x} = f(z).$$

$$\frac{\partial F(z)}{\partial y} = i f(z).$$

In this is so $F(z)$ fulfills the cauchy Riemann equation.

$$\frac{\partial F}{\partial x} = -i \frac{\partial F}{\partial y}.$$

Since $f(z)$ is by assumption continuous (otherwise $\int_\gamma f\,dz$ would not be defined) $F(z)$ is analytic with the derivative $f(z)$.

The integral $\int_\gamma f\,dz$, with continuous f depends only on the end points of γ iff f is the derivative of an analytic function on Ω.

Defn: cauchy's theorem for a Rectangle:

We consider specifically a rectangle R defined by inequalities $a \leq x \leq b$, $c \leq y \leq d$.

It's perimeter can be considered as a simple closed curve consisting of four line segments whose direction.

We choose so that R lies to the left of the directed segments.

The order of the vertices is [illegible] (b,d), (a,d) . we refer to this closed [illegible] as the boundary curve (or) contour of R and we denote it by ∂R.

Theorem: (cauchy's theorem)

If the function $f(z)$ is analytic on R. then $\int_{\partial R} f(z)\,dz = 0$.

proof:

Let $\int_{\partial R} f(z)\,dz = 0$ attain.

Let $\eta(R) = \int_{\partial R} f(z)\,dz$.

=> which we will also use for any rectangle contained in the given one.

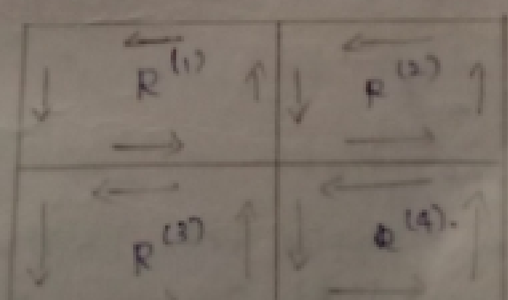
=> If R is divided into four congruent rectangles.

$R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}$. we find that

$\eta(R) = \eta(R^{(1)}) + \eta(R^{(2)}) + \eta(R^{(3)}) + \eta(R^{(4)})$ →①

=> for the integrals over the common sides cancel each other.

Draw the diagram:



=> It follows from ① that at least one the rectangles $R^{(k)}$, $k = 1, 2, 3 \ldots$

Satisfy the condition

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(54) $|\eta(R^{(k)})| \geq \frac{1}{4} |\eta(R_{n-1})|$

a sequence of the rectangle

$R > R_1 > R_2 > \dots R_n > \dots$

with the property

$|\eta(R_n)| \geq \frac{1}{4} |\eta(R_{n-1})|.$

$|\eta(R_n)| \geq 4^{-n} |\eta(R)|$ — (1)

The rectangles R_n converges to a point $z^* \in R$.

Such that

R_n will be contained in neighbourhood of $|z - z^*| < \delta$.

we choose δ so small that $f(z)$ is analytic in

$|z - z^*| < \delta.$

if $\varepsilon > 0$ is given by we can choose

$$\left| \frac{f(z) - f(z^*)}{z - z^*} - f'(z^*) \right| < \varepsilon.$$

$$|f(z) - f(z^*) - (z - z^*) f'(z^*)| < \varepsilon |z - z^*| \quad \text{— (2)}$$

for

$|z - z^*| < \delta$

we assume that

δ satisfies both conditions and that R_n is contained in $|z - z^*| < \delta$.

i.e) $\int_{\partial R_n} dz = 0$

$\Rightarrow \int_{\partial R_n} z \, dz = 0.$

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by using the eqn of this form

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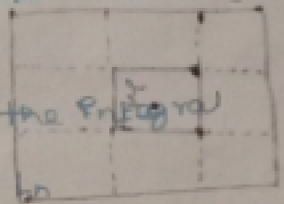
$$\eta(R_n) = \int_{\partial R_n} |f(z) - f(z^*) - (z-z^*) f'(z^*)| \, dz.$$

By using this eqn ③

$$|\eta(R_n)| \leq \varepsilon \int_{\partial R_n} |z-z^*| \, |dz|. \rightarrow ④$$

In the last integral $|z-z^*|$ is atmost equal to the length of the diagonal of R_n

$\Rightarrow$ In d_n denote the length of the diagonal of R_n.



And hence the integral

$R_n \leq d_n \, L_n$

$\Rightarrow$ But if d and L of the corresponding candidates for the original rectangle R.

It is clear that

$$\left. \begin{array}{l} d_n = 2^{-n} d \text{ and} \\ L_n = 2^{-n} L. \end{array} \right\} \rightarrow ⑤$$

The eqn of ⑤ is sub in eqn ④.

and hence $|\eta(R_n)| \leq 4^{-n} d L \varepsilon$.

and comparision of eqn ③ is

$|\eta(R)| \leq d L \varepsilon$.

Since ε is arbitary.

Therefore

$\eta(R) = 0$.

Theorem:

Let $f(z)$ be analytic on the set R' obtained from a rectangle R by omitting a finite no of interior point ξ_j.

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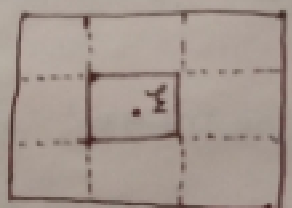
if it is prove that $\lim_{z \to \xi_j} (z - \xi_j) f(z) = 0 \quad \forall j.$

then $\int_{\partial R} f(z)\, dz = 0.$

Proof.

Let $f(z)$ be analytic on the set R' obtained from the rectangle.

Draw the diagram:



it is sufficient to consider the case of the single exponential point ξ.

for using the divided into smaller rectangles of R which contains atmost one ξ_j.

Id the R_0 in the center of roctande R. it is corresponding the equation of the functions $f(z)$ is analytic on are

then $\int_{\partial R} f(z)\, dz = 0. \longrightarrow ①.$

we obtained the after cancellation

$\int_{\partial R} f dz = \int_{\partial R_0} f dz \rightarrow ②$

If $\varepsilon > 0$ we can choose the rectangle R_0. So small that

$|z| \leq \varepsilon / |z - \xi|$ on R. $\rightarrow ③$

by the eqn ③ sub in ②.

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we obtain

$$\left| \int_{\partial R} f\,dz \right| \leq \varepsilon \int_{\partial R_0} \frac{|dz|}{|z-\xi|}.$$

That is R_0 is a square of centre ξ, elementary estimate show that

$$\int_{\partial R} \frac{|dz|}{|z-\xi|} < 8$$

Thus we obtain

$$\left| \int_{\partial R} f\,dz \right| < 8\varepsilon.$$

Since ε is arbitrary

Therefore $\int_{\partial R} f(z)\,dz = 0.$

Cauchy's theorem in a Disk:

It is not true that [the integral of an analytic function over a closed curve is always zero ~~Indeed~~

~~we have found that~~ $\int_C \frac{dz}{z-a} = 2\pi i$

when c is circle about a.

proof:

In order to make sure that the integral vanishes, it is the necessary to make a special assumption concerning the region in which $f(z)$ is known to be analytic and to which curve γ is restricted.)

=> we are not yet in a position to formulate this condition, and for this reason.

We must restrict attention to a very special case.

$\Rightarrow$ In what follows we assume that is an open disk $|z - z_0| < \rho$ to be denoted by Δ.

$\int_{\partial R_0} \frac{|dz|}{|z - z_0|} < \varepsilon$ c is constant.

$\Rightarrow \int_{\partial R_0} \frac{|dz|}{|z - z_0|} \Rightarrow \int \frac{dz}{z - a} = 2\pi i$

Theorem:

If $f(z)$ is analytic in an open disk Δ, then $\int_\gamma f(z)\,dz = 0$ for every closed curve γ in Δ.

Proof:

Let $\int_\gamma f(z)\,dz = 0. \longrightarrow (1)$

we define a function $F(z)$ by

$F(z) = \int_\sigma f\,dz. \longrightarrow (2)$

$\Rightarrow$ where σ consists of the horizontal line segment from the centre (x_0, y_0) to (x, y_0) and the vertical segment from (x, y_0) to (x, y).

$\Rightarrow \frac{\partial F}{\partial y} = i f(z) \longrightarrow (3)$

$\Rightarrow$ This choice defines the same function $f(z)$, and we obtain $\frac{\partial F}{\partial x} = f(z)$.

$\Rightarrow F(z)$ is analytic in Δ with the derivative $f(z)$, and $f(z)dz$ is an exact differential.

$\Rightarrow$ The rectangle with the opposite vertices z_0 and z as soon as it contains z.

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⇒ A rectangle a half plane or the inside of an ellipse.

Therefore $\int_\delta f(z)dz = 0$ where $f(z)$ is analytic

every closed curve in triangle Δ.

Therefore $\int_\gamma f(z)dz = 0$.

Theorem: 2.5

Let $f(z)$ be analytic in the region Δ' obtained by omitting a finite number of points ξ_j from an open disk Δ.

If $f(z)$ satisfies the conditions

$$\lim_{z \to \xi_j} (z - \xi_j) f(z) = 0 \quad \forall j.$$

then $\int f(z)\, dz = 0$.

holds for any closed curve γ in Δ'.

proof:

Draw the diagram

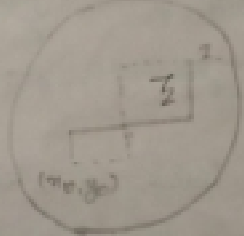

fig (1).

⇒ Let σ pass through the exceptional point.

⇒ Assume first that no ξ_j lies on the lines $x = x_0$ and $y = y_0$.

⇒ It is then possible to avoid the exceptional points by letting σ consist of three segments. (fig (1)).

$$\int_\gamma f(z)\, dz = 0 \longrightarrow ①.$$

⇒ Then case there are exceptional points on the lines $x = x_0$ and $y = y_0$.

⇒ Therefore four line segments in the

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Place of three.

Hence the proof.

The index of a point with respect to a closed curve:

Lemma: 1

If the piecewise differentiable closed curve γ does not pass through the point a then the value of the integral $\int_\gamma \frac{dz}{z-a}$ is a multiple of $2\pi i$.

proof:

This lemma may seem trivial, for we can write

$$\int_\gamma \frac{dz}{z-a} = \int_\gamma d \log(z-a) = \int_\gamma d \log|z-a| + i \int_\gamma d \arg(z-a).$$

=> when z describes a closed curve, $\log|z-a|$ return to its initial value and $\arg(z-a)$ increases or decreases by a multiple of 2π.

=> If the equation of γ is $z = z(t)$, $\alpha \le t \le \beta$, Let us consider the function

$$h(t) = \int_\alpha^t \frac{z'(t)}{z(t)-a} dt$$

=> It is defined and continuous on the closed curve interval $[\alpha, \beta]$, and it has the derivative.

$$h'(t) = \frac{z'(t)}{z(t)-a}.$$

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whenever $z'(t)$ is continuous.

$\Rightarrow$ a finite number of points and since the function is continuous it must reduce to a constant we have thus

$$e^{h(t)} = \frac{z(t) - a}{z(\alpha) - a}$$

Since $z(\beta) = z(\alpha)$

we obtain $e^{h(\beta)} = 1$.

and therefore $h(\beta)$ must be a multiple of $2\pi i$.

2

Defn: Index of the point:

The index of the point a with respect to the curve γ by the equation (see the diagram, facing page)

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$$

Defn: Winding number:

The index is also called the winding number of γ with respect to a.

It is clear that $n(-\gamma, a) = -n(\gamma, a)$.

properties of consequence:

(i) If γ lies between inside of a circle, then $n(\gamma, a) = 0$ for all points a outside of the same circle.

(ii) As a function of a the index $n(\gamma, a)$ constant in each of the regions determined by γ and zero in the unbounded region.

$$\int_\gamma \left(\frac{1}{z-a} - \frac{1}{z-b} \right) dz = 0.$$

Lemma: 2

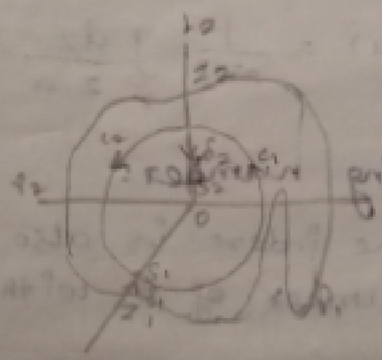
Let z_1, z_2 be two points on a closed curve γ. which does not pass through the origin denote the subarc from z_1 to z_2 in the direction of the curve by γ_1 and the subarc from z_2 to z_1 by γ_2.

Suppose that z_1 lies in the lower half plane and z_2 in the upper half plane. If γ_1 does not meet the negative real axis and γ_2 does not meet the positive real axis.

then $n(\gamma, 0) = 1$.

Proof:

Draw the diagram. 3



$\Rightarrow$ The half lines L_1 and L_2 from the origin through z_1 and z_2.

$\Rightarrow$ Let S_1, S_2 be the points in which L_1, L_2.

intersect a circle c about the origin

$\Rightarrow$ If c is described in the positive sense, the arc c_1 from S_1 and S_2 does not intersect the positive axis.

$\Rightarrow$ Denote the directed line segment from z_1 to S_1 and from z_2 to S_2 by δ_1, δ_2

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$\Rightarrow$ introducing the closed curves

$\sigma_1 = \gamma_1 + \delta_2 + \delta_2 - c_1 - \delta_1$,

$\sigma_2 = \gamma_2 + \delta_1 - c_2 - \delta_2$

we find that

$n(\gamma, 0) = n(c, 0) + n(\sigma, 0) + n(\sigma_2, 0)$

because of cancellations.

$\Rightarrow$ Hence the origin belong to the unbounded region determined by σ_1, and

we obtain $n(\sigma_1, 0) = 0$

$\Rightarrow$ For a similar reason $n(\sigma_2, 0) = 0$

and we conclude that $n(\gamma, 0) = n(c, 0) = 1$.

4

The integral formula:

Let $f(z)$ be analytic in an open disk

Triangle consider the closed curve γ in Triangle and a point $a \in \Delta$.

which does not lie on γ.

we apply the Gauss theorem to the function.

$$F(z) = \frac{f(z) - f(a)}{z - a}.$$

This fn is analytic for $z \neq a$.

for $z = a$ it is not defined but it satisfy the condition

$$\lim_{z \to a} F(z)(z-a) = \lim_{z \to a} (f(z) - f(a)) = 0$$

We conclude that $\int_\gamma \frac{f(z) - f(a)}{z - a} dz = 0$

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This eqn can be written in the form

$$\int_\gamma \frac{f(z)}{z-a}\,dz = f(a)\int_\gamma \frac{dz}{z-a}.$$

The integral in the right hand side is by defns $2\pi i\ n(\gamma, a)$

State and prove cauchy's integral formula:

Statement:

If $f(z)$ is analytic in an open disk Δ, and let γ be a closed curve in Δ. for any point A not on γ

$$n(\gamma, a) \cdot f(a) = \frac{1}{2\pi i}\int \frac{f(z)\,dz}{z-a}$$

where

$n(\gamma, a)$ is the winding index of a with respect to γ

proof:

Let $f(z)$ is analytic in Δ.

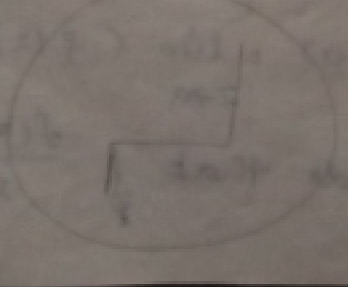
i.e) $n(\gamma, a) \cdot f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)\,dz}{z-a} \rightarrow (1)$
for the case that

If ~~that case~~ for a is not in Δ from the eqn (1).

and let ~~Ontot~~ $n(\gamma, a)$ and the integral in the right hand member are both zero

by using theorem 5

Draw the diagram



65) let $f(z)$ be a analytic in region a finite number of ζ_j

from an open disk δ.

If $f(z)$ satisfy the conditions

Then $\int_{\gamma} f(z)\,dz = 0. \rightarrow (2)$

we can prove that let σ pass through the exponential points.

and ζ_j lies on the lines $x = x_0$ and $y = y_0$

=> There are the exponential points on the lines of (2)

$x = x_0$ and $y = y_0$.

The most common application to the case where $n(\gamma, a) = 1$

we have

then $f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)\,dz}{z-a} \rightarrow (3)$

This equation (3) is called representation formula.

=> Then the value of $f(a)$ as the values of $f(z)$ on γ are given.

=> Together with the fact the $f(z)$ is analytic in Triangle.

=> This is the eqn (3) and let a take different values

=> That the order of a with respect to γ remains equal to 1.

Then the value of a its ei

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convenient to change the notation and

write the eqn in the form

$$f(z)=\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)\,d\xi}{\xi - z} \to (4)$$

$\Rightarrow$ we must remember that it is

only $n(\gamma,a)=1$

and that we have prove it only when $f(z)$ is analytic in disk.

State and prove Jordan curve:

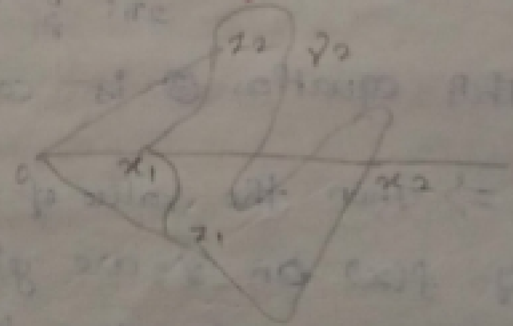
Stmt:

If every Jordan curve every plane determines exactly two regions. The index the winding number γ with respect to a

$n(-\gamma,a) = -n(\gamma,a)$

Proof:

Draw the diagram



Let the index of the point a with respect to the curve γ by the equation of

$$n(\gamma,a)=\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} \to (1)$$

$$\Rightarrow n(-\gamma,a) = -n(\gamma,a) \to (2)$$

There exist a Jordan curve γ has atleast two formulate compo-

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$\Rightarrow$ there exist a point a with $n(\gamma, a) \neq 0$.

$\Rightarrow$ we may assume that

$\operatorname{Re} z > 0$ or γ

and the point

There are the point $z_1, z_2 \in \gamma$ with

$\operatorname{Im} z_1 < 0$, $\operatorname{Im} z_2 > 0$.

z_2 which is the point of γ on the line segment from 0 to z_2 and from 0 to z_2.

$\Rightarrow$ and γ_1 and γ_2 be the arcs are γ for z_1 to z_2

$\Rightarrow$ let σ_1 be a closed curve that consisting of the line segment.

followed by γ_1 and the segment from z_2 to zero

$\Rightarrow$ and let σ_2 be constructed in the same way with γ_2 in the plane of γ_1.

$\therefore \sigma_1 - \sigma_2 = \gamma$ or $-\gamma$

Therefore $n(-\gamma, a) = -n(\gamma, a) \rightarrow (1)$

Let the given can of the highes derivatives of the formula

i.e) $f(z) = \dfrac{1}{2\pi i} \displaystyle\int_{\gamma} \dfrac{f(\xi)\, d\xi}{\xi - z} \rightarrow (1)$

$\Rightarrow$ a function $f(z)$ which is analytic in an arbitary region Ω.

$\Rightarrow$ To a point $a \in \Omega$ be determined

$\Rightarrow \delta$-neighbourhood Δ contained in

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and in Δ of a circle c about a

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$\Rightarrow$ Since $n(c,a)=1$

we have

$n(c,z)=1$.

for all the points z inside of c

for such z we obtain

$$f(z)=\frac{1}{2\pi i}\int_c \frac{f(\zeta)\,d\zeta}{\zeta - z} \rightarrow (2)$$

the integral can be differentiated under the sign of integration from eqn (2)

we get

$$f'(z)=\frac{1}{2\pi i}\int_c \frac{f(\zeta)\,d\zeta}{(\zeta - z)^2} \rightarrow (3)$$

and

$$f^n(z)=\frac{n!}{2\pi i}\int_c \frac{f(\zeta)\,d\zeta}{(\zeta - z)^{n+1}} \rightarrow (4)$$

Since every point in Ω lies inside of some such ~~search~~ circle the extens will be proved in the region Ω.

Theorem:

Statement:

If $\phi(\zeta)$ is continuous on the arc γ then the function $F_n(z)=\int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta - z)^n}$ is analytic in each of the regions determined by γ and its derivative is $F'_n(z)=nF_{n+1}(z)$

proof:

Let $F_1(z)$ is continuous.

Let z_0 be a point not on γ and choose the neighbourhood $|z-z_0|<\delta$

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so that it does not meet γ.

69 By restricting z to the smaller neighbourhood $|z-z_0| < \delta/2$

we claim that

$|\zeta - z| > \delta/2 \ \forall\ \zeta \in \gamma$

$$F_1(z) - F_1(z_0) = (z-z_0)\int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta - z)(\zeta - z_0)}.$$

we obtain

$$|F_1(z) - F_1(z_0)| < |z-z_0|\,.\,\frac{2}{\delta^2}\int_\gamma |\phi||d\zeta|$$

and this inequality proves the continuity of $F_1(z)$ at z_0.

From this part of the lemma

Applied the function $\phi(\zeta)/(\zeta - z_0)$.

we conclude that the differential quotient

$$\frac{F_1(z) - F_1(z_0)}{z-z_0} = \int_\gamma \frac{\phi(\zeta)\,d\zeta}{(\zeta - z)(\zeta - z_0)}.$$

Tends to the limit $F_2(z_0)$ as $z \to z_0$

hence it prove that $F_1'(z) = F_2(z)$.

The general case is proved by Induction

Suppose we have shown that

$F'_{n-1}(z) = (n-1)F_n(z)$

from the identity

$F_n(z) - F_n(z_0)$

$$= \left[\int_\gamma \frac{\phi\,d\zeta}{(\zeta - z)^{n-1}(\zeta - z_0)} - \int \frac{\phi\,d\zeta}{(\zeta - z_0)^n}\right] + (z-z_0)\int_\gamma \frac{\phi\,d\zeta}{(\zeta - z)^n(\zeta - z_0)}$$

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by the Rij induction hypothesis applied to $\phi(z)/(\zeta - z_0)$

The first term tends to zero for $z \to z_0$ and in the second term the factor of $z - z_0$ is bounded in a neighbourhood of z_0.

If we divided the identity by $z - z_0$ and let z tends to z_0, the quotient in the first term tends to a derivative which by induction hypothesis equals $(n-1) F_{n+1}(z_0)$ and has the limit $F_{n+1}(z_0)$.

hence $F'_n(z_0)$ exists and equal $n F_{n+1}(z_0)$

Liouville's theorem:

Statement:

A function which is analytic and bounded in the whole plane must reduce to a constant.

Proof:

Use of a simple estimate derived from

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z)^{n+1}} \longrightarrow (1)$$

Let the radius of C be r, and assume that

$|f(\zeta)| \leq M$ on C.

we apply eqn (1) with $z = a$.

we obtain

$$|f^{(n)}(a)| \leq M n! \cdot r^{-n} \longrightarrow (2)$$

For Liouville's theorem, we need only the case $n = 1$

The hypothesis means that

$|f(\zeta)| \leq M$ on all circles.

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hence we let λ tends to ∞ and α_n ω leads to $f'(a) = 0$ $\forall a$.

we conclude that the function is constant.

71 State and prove Morera's theorem.

Stmt:

If $f(z)$ is defined and continuous in a region Ω and if $\int_\gamma f\,dz = 0$. for all closed curve γ in Ω. Then $f(z)$ is analytic in Ω.

Proof:

Let $f(z)$ is defined and continuous in a region Ω

i.e) $\int_\gamma f\,dz$ and defined $F(z)$ is analytic with the derivative $f(z)$.

by using the defn of the integral $\int f\,dz$ with continuous f.

depends only on the endpoints of γ if and only if f is the derivative of and analytic function Ω.

$\Rightarrow$ $f(z)$ is itself analytic.

i.e) $\int_\gamma (z-a)^n dz = 0$ $\forall$ closed curve, (definition) Provided that the integer n is $n \geq 0$.

$\Rightarrow$ $f(z)$ is the derivative of an analytic function $F(z)$.

where $f(z)$ is then itself analytic

i.e) $\int_\gamma f\,dz = 0$.

Therefore $f(z)$ is analytic on Ω.

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Fundamental theorem of algebra:

72 Stmt:

A function which is analytic and bounded in the whole plane must reduce to a constant.

proof:

Let $f(z)$ is defined and continuous in a region Ω.

Suppose that $p(z)$ is a polynomial of a degree greater than >0.

$\Rightarrow$ If $p(z)$ where never 0

$\Rightarrow$ The function $1/p(z)$ would be analytic in the whole plane.

$\Rightarrow$ We know that $p(z) \to 0$ for $z \to \infty$

There $1/p(z) \to 0$.

by using Liouville thm $1/p(z)$ would be constant.

i.e) Then the equation $p(z) = 0$.

Stmt 2m 5c cauchy's estimate:

Let the eqn $|f^{(n)}(a)| \leq |M_n| r^{-n} \to ①$

This is the eqn ① is called the cauchy's estimate.

proof:

Let $|f(z)| \leq M$ on C

with $z = a$.

The all the derivative of an analytic function cannot be as

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$\Rightarrow$ There must always exists and M and, it is using the eqn. its eqn fulfilled.

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c) The objective of the inequality r be choosen the object beginning to minimize the function

$~~f(r)~~ = M(r)\, r^{-1}$

where $M(r)$ is the maximum of $|z-a| = r$

Therefore

$|f^{(n)}(a)| \leq |M_n|\, r^{-n}$

Theorem:

Let the function $\phi(z,t)$ be continuous as a function both variables when z lies in a region Ω. and $\alpha \leq t \leq \beta$.

Supposed further such that that $\phi(z,t)$ is analytic as a function of $z \in \Omega$.

for any fixed $~~\alpha~~ \beta$, then

$F(z) = \int_{\alpha}^{\beta} \phi(z,t)\, dt$ is automatic at z.

and $F'(z) = \int_{T}^{R} \frac{\partial \psi(z,t)}{\partial z}\, dt.$

Proof:

Let the function $\phi(z,t)$ is continuous and variables z lies on region Ω and $\alpha \leq t \leq B$.

analytic

To prove

This represent $\phi(z,t)$ as a cauchy integral

$\phi(z,t) = \frac{1}{2\pi i} \int_{c} \frac{\cancel{\phi(\xi, z)}}{} \frac{\phi(\xi, z)}{\xi - z}\, dz \rightarrow$

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fill in the necessary details to obtain

$$F(z) = \int_0 \left(\frac{1}{2\pi i} \int_\alpha^\beta \varphi(z, t)\, dt \right) \frac{d\xi}{\xi - z} \rightarrow (2)$$

where

$$\varphi(z, t)\, dt = \left(\frac{1}{2\pi i} \int_\alpha^\beta \phi(\xi, t)\, dt \right) \frac{d\xi}{\xi} =$$

The partial derivative of differentiation on equation (2) is we get

$$F'(z) = \int_\alpha^\beta \frac{\partial \phi(z, t)}{\partial z}\, dt \rightarrow (3)$$

Therefore equation (2) and (3) ...

hence the result of the following theorem.

UNIT-III

Local properties of analytic functions:

We have already proved that an analytic function has derivatives of all order.

In this section we will make a closer study of the local properties.

It will include a classification of the isolated singularities of analytic functions.

Removable singularities : Taylor's theorem:

Singularity: A point $z = a$ is said to be singularity of a function $f(z)$, if $f(z)$ not analytic at $z = a$.

Removable Singularity:

A point $z = a$ is said to be removable singularity of a function $f(z)$, if $\lim_{z \to a} f(z) =$ finite. It is called a removable singularity.

Ex:

$f(z) = \frac{\sin z}{z} = \sin z$

$\lim f(z) = \cos z / 1 = \cos 0 /$

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Theorem:

Suppose that $f(z)$ is analytic in the region Ω' obtaining a point from a region Ω. A necessary and sufficient condition that there exists an analytic function in Ω which coincides with $f(z)$ in Ω' is that

$\lim_{z \to a} (z-a) f(z) = 0$ the extended function is uniquely determined.

Proof: The necessary part.

The necessity and the uniqueness are trivial. Since the extended function must be continuous at a.

To prove that

the sufficiency we draw a circle c about a so that c and its inside are contained in Ω.

by using cauchy's formula is valid

and we can write

$$f(z) = \frac{1}{2\pi i} \int_c \frac{f(\xi)\,d\xi}{\xi - z}$$

for all $z \neq a$ inside of c.

But the integral in the right hand member represents an analytic function of z throughout the inside of c.

Consequently the function which is equal to $f(z)$ for $z \neq a$ and which has the value

$$\frac{1}{2\pi i} \int_c \frac{f(\xi)\,d\xi}{\xi - a} \longrightarrow (1)$$

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natural to denote the extended function by $f(z)$ and the value (5) by $f(a)$.

we apply this result to the function

76 $$F(z) = \frac{f(z) - f(a)}{z - a}$$

used in the proof of cauchy's formula

It is not defined for $z = a$.

But It satisfies the condition

$$\lim_{z \to a} (z - a) F(z) = 0.$$

$\Rightarrow$ The limit of $F(z)$ as $z \to a$ is $f'(z)$

$\Rightarrow$ Hence there exist an analytic function.

which is equal to

$F(z)$ for $z \neq a$

and equal to $f'(a)$ for $z = a$.

$\Rightarrow$ Let us denote the this function by $f_1(z)$

$\Rightarrow$ Repeating the process we can define an analytic function $f_2(z)$.

which is equal to

$f_1(z) - f_1(a)/(z - a)$ for $z \neq a$

and

$f_1'(a)$ for $z = a$.

$\Rightarrow$ Let $f_n(z)$ is defined can be written in the form.

$$f(z) = f(a) + (z - a) f_1(z)$$

$$f_1(z) = f_1(a) + (z - a) f_2(z).$$

$$- \quad - \quad - \quad -$$

$$- \quad - \quad - \quad -$$

$$f_{n-1}(z) = f_{n-1}(a) + (z - a) f_n(z)$$

valid also for $z = a$.

we obtain

$$f(z) = f(a) + (z-a) f_1(a) + (z-a)^2 f_2(a) + \cdots + (z-a)^{n-1} f_{n-1}(a) + (z-a)^n f_n(z).$$

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determine with the differentiating n times and setting $z = a$.

we find

$$f^{(n)}(a) = n!\, f_n(a).$$

Theorem: (Taylor's theorem).

If $f(z)$ is analytic in a region Ω, containing a, it is possible to write

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + f_n(z)(z-a)^n.$$

where $f_n(z)$ is analytic in Ω.

Proof:

Let $f(z)$ is analytic in a region Ω. containing a.

i.e)

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + f_n(z)(z-a)^n \rightarrow (1).$$

by using eqn (1) at analytic $f_n(z)$ has a expression as a line integral

$\Rightarrow$ using the same circle C as before we have first.

$$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f_n(\xi)\, d\xi}{\xi - z}.$$

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for of $f_n(\xi)$.

we sub the expression of the eqn ①.

$\Rightarrow$ there will be one main term containing

 $f(\xi)$

$\Rightarrow$ the remaining terms are except for the constant factors of the form.

$$F_\nu(a) = \int_c \frac{d\xi}{(\xi - a)^\nu (\xi - z)}, \quad \nu \geq 1.$$

But

$$F_1(a) = \frac{1}{z-a} \int_c \left(\frac{1}{\xi - z} - \frac{1}{\xi - a} \right) dz = 0.$$

Identically for a inside of C.

by using lemma: 3

$$F_{\nu+1}(a) = F_1^{(\nu)}(a) / \nu!$$

and thus $F_\nu(a) = 0 \;\forall\; \nu \geq 1$.

hence the equation for $f_n(z)$ reduces to eqn ⓐ.

$$f_n(z) = \frac{1}{2\pi i} \int_c \frac{f(\xi)\, d\xi}{(\xi - a)^n (\xi - z)} \longrightarrow ②.$$

The representation valid inside of c.

Zeros and poles:

If $f(z)$ and is a analytic in region Ω. That is $f(z) = f(a) + \frac{f'(a)}{1!}(z-a) +$

$$\frac{f''(a)}{2!}(z-a)^2 + \dots + \frac{f^{n-1}(a)}{(n-1)!}(z-a)^{n-1}$$

$$+ f_n(z)(z-a)^n \longrightarrow ①$$

and all derivatives $f^{(\nu)}(a)$ vanish.

i.e) $f(z) = f_n(z)(z-a)^n \longrightarrow ②$

$$f_n(z) = \frac{1}{2\pi i}\int_c \frac{f(\xi)\,d\xi}{(\xi-a)^n(\xi-z)} \rightarrow (3).$$

the disk with the circumstance c has to contained in the region Ω.

79 In which $f(z)$ is definite and analytic.

$\Rightarrow$ the absolutely value $|f(z)|$ has maximum on M on c. with radius c or is the denoted by R, we defined

$$|f_n(z)| \leq \frac{M}{R^{n-1}(R-|z-a|)}$$

by using in eqn (2)

$$|f(z)| \leq \left(\frac{|z-a|}{R}^n\right) \cdot \frac{MR}{R-|z-a|}$$

But $(|z-a|) < R$.

Hence $f(z)=0$ inside of c.

Isolated singularity:

If $z=a$ is a singular point of function $f(z)$ and if there exists a neighbourhood of $z=a$. containing no other singular points of $f(z)$.

Then $z=a$ is said to be isolated singularity of $f(z)$.

Ex:

$$f(z) = \frac{1}{((z-1)(z-3)}$$

i.e) $z=1$, $z=3$

Non isolated singularity:

If $f(z)$ has infinite of singular point in every neighbourhood of z ...

Then $z=a$ is a non isolated singular point.

Also it is a limit point of the set of all singular point of $f(z)$.

Ex:

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$$\frac{1}{\sin z} = 0$$

$$\frac{1}{\sin(0)} = 1/0 = \infty.$$

pole:

If $f(z)$ has infinite no of singular point in every neighbourhood of analytic function

$$\boxed{\lim_{z\to a} f(z) = \infty}$$

It is called a pole.

essential singular point: (singularity)

Let $f(z)$ be a analytic function.

i.e) $\lim_{z\to a} f(z)$ does not exists

Ex:

$f(z) = e^{1/z}$

Soln:

$$e^{1/z} = 1 + 1/z + \frac{1}{2!z^2} + \frac{1}{3!}z^3 + \dots$$

$z=0$.

$$e^{1/0} = 1 + 1/0 \dots = \infty.$$

e

$$\left.\begin{array}{l} e^z = \infty \\ \sin z = \infty \end{array}\right\} \text{essential singularity.}$$

Theorem:

State and prove weierstrass theorem:

An analytic function comes arbitrarilly close to any complex value in every neighbourhood

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of an essential singularity.

proof:
=> let a complex number A and a $\delta > 0$ such that

$|f(z) - A| > \delta$

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in a neighbourhood of a (except for $z = a$)

=> for any $\alpha < 0$ we have then

$\lim_{z \to a} |z - a|^{\alpha} |f(z) - A| = \infty$.

=> ~~Hence~~ Hence a would not be an essential singularities of $f(z) \to A$.

=> Accordingly there exists a β with

$\lim_{z \to a} |z - a|^{\beta} |f(z) - A| = 0$.

and we ~~are~~ free to choose $\beta > 0$.

=> Since in that case

$\lim_{z \to a} |z - a|^{\beta} |A| = 0$

it would follow that

$\lim_{z \to a} |z - a|^{\beta} |f(z)| = 0$.

and a would not be as essential singularity of $f(z)$.

=> The notion of isolated singularity applies also to functions which are analytic in a neighbourhood $|z| > R$ of ∞.

=> Since $f(\infty)$ is not defined we treat ∞ as an isolated singularities.

=> And by convention it has the same character of removable singularity pole or essential singularity as the singularity of

$g(z) = f(1/z)$ at $z = 0$.

=> If the singularity is non essential $f(z)$ has

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an algebraic order h such that

$\lim_{z \to \infty} z^{-h} f(z)$ is neither zero nor infinity

$\Rightarrow$ and for a pole the singular part is a polynomial in z.

82 $\Rightarrow$ If ∞ is an essential singularity.

The local mapping :-

A function $f(z)$ if it is analytic and not identically zero in an open disk Δ.

Let γ be a closed curve in Δ such that $f(z) \neq 0$ on γ.

i.e) we write $f(z) = (z-z_1)(z-z_2)\dots(z-z_n)\, g(z)$

we obtain $g(z)$ is analytic and $\neq 0$ in, logarithmic derivatives.

$$\frac{f'(z)}{f(z)} = \frac{1}{z-z_1} + \frac{1}{z-z_2} + \dots + \frac{1}{z-z_n} \; \frac{g'(z)}{g(z)}$$

for $z \neq z_j$ and particularly on γ.

Since $g(z) \neq 0$ in Δ.

by using cauchy's theorem,

$$\int_\gamma \frac{g'(z)}{g(z)}\, dz = 0$$

by using the definition on we find

$$n(\gamma, z_1) + n(\gamma, z_2) + \dots + n(\gamma, z_n)$$
$$= \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz$$

Theorem: 10 (Bolzano weirstrass theorem)

Statement :-

Let z_j be the zeros of a function

83 $f(z)$. which is analytic in a disk Δ and does not vanish Identically, each zero being counted as many times as it's order indicate.

for every closed curve γ in Δ which does not passes through the zero.

$$\sum_j n(\gamma, z_j) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz$$

where the some has only a finite number of terms not equal to zero.

Proof:

Let z_j be the zeros of a function $f(z)$. which is analytic in a disk Δ and does not vanish identically.

The function $w = f(z)$ maps γ on to a closed curve Γ in the w-plane and we find

$$\int_\Gamma \frac{dw}{w} = \int_\gamma \frac{f'(z)}{f(z)} dz$$

the total number of zero

$$\left[\text{i.e.) } \sum_j n(\gamma, z_j) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz \right] \rightarrow (1)$$

The formula for eqn (1) has thus the following interproduction.

$$n(\Gamma, 0) = \sum_j n(\gamma, z_j) \rightarrow (2)$$

Case (i)

$z = 0$.

By using the eqn (1)

$n(\gamma, z_j) = 0$

Case (ii)

If $z = 1$. By using eqn (1)

$n(\gamma, z_j) = 1$

The using the eqn ⓑ total number of zero n closed by γ.

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i.e.) In the case (i) and (ii) γ is circle.

Let a be an arbitary complex valued. By using in this theorem to $f(z) = z$.

$\Rightarrow$ The zeros of $f(z) - a$ are the roots of the eqn $f(z) = a$.

and we denoted them by $z_j(a)$.

$\Rightarrow$ In the place of eqn ⓐ we obtain the formula.

$$\sum_j n(\gamma, z_j(a)) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - a} dz$$

$\Rightarrow$ In the place of eqn ⓑ takes the form

$$n(\Gamma, a) = \sum_j n(\gamma, z_j(a))$$

It is necessirily to assume that $f(z) \neq a$ on γ.

If a and b are in the same region determined by Γ.

we know that $n(\Gamma, a) = n(\Gamma, b)$.

we have also

$$\sum_j (n(\gamma, z_j(a))) = \sum_j n(\gamma, z_j(b))$$

$\Rightarrow$ If γ is circle it is follows that $f(z)$ takes the value a and b equally many times inside of γ.

Theorem: 11 (iif)

Suppose that $f(z)$ is analytic at z_0, $f(z_0) = w_0$ and that $f(z) - w_0$ has a zero of order n at z_0.

If $\varepsilon > 0$ is sufficiently small there exists a corresponding $\delta > 0$ $\ni$ for all a with $|a - w_0| < \delta$. The eqn $f(z) = a$ has exactly n roots in the disk $|z - z_0| < \varepsilon$.

proof:

35 $\Rightarrow$ we can choose ε show that $f(z)$ is defined and analytic for $|z - z_0| \leq \varepsilon$

and show that z_0 is the only zeros of $f(z) - w_0$ in this disk.

$\Rightarrow$ let γ be the circle $|z - z_0| = \varepsilon$ and Γ it's image under the mapping $w = f(z)$.

$\Rightarrow$ Since w_0 belongs to the complement of the closed set Γ.

There exists a neighbourhood $|w - w_0| < \varepsilon$.

which does not intersects.

$\Rightarrow$ Let is follows immediately that all the values in this neighbourhood a are taken the same number of times inside of γ. The eqn $f(z) = w_0$ has exactly n.

Co-inside roots inside of γ.

And hence every value a is taken n times.

$\Rightarrow$ It is understood that multiple roots are ~~contor~~ counter according to the multiple factor but if ε is sufficiently small.

we can assume that the all the roots of the equations $f(z) = a$ are simple for $a \neq w_0$.

$\Rightarrow$ It is sufficient to choose ε;

So that $f'(z)$ does not vanish for $0 < |z - z_0| < \varepsilon$.

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Corollary : 1

A non-constant analytic function maps open sets onto open sets.

Proof:

This is merely another way of saying that the image of every sufficiently small disk

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$|z - z_0| < \varepsilon$

contains a neighbourhood

$|w - w_0| < \delta$

$\Rightarrow$ In the case $n = 1$ there is one-to-one correspondence between the disk

$|w - w_0| < \delta$

and an open subset Δ of $|z - z_0| < \varepsilon$.

$\Rightarrow$ Since open sets in the z-plane correspond to open sets in the w-plane the inverse function on $f(z)$ is continuous and the mapping is topological.

$\Rightarrow$ The mapping can be restricted to a neighbourhood of z_0 contained in Δ.

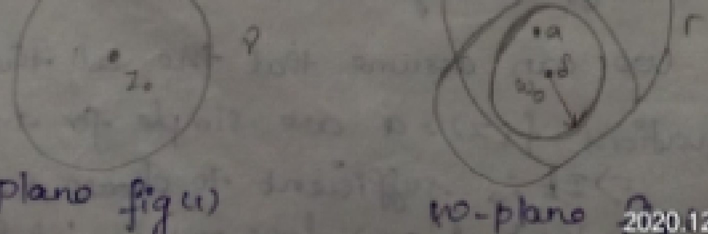
and we are able to state.

Corollary : 2

If $f(z)$ is analytic at z_0 with $f'(z_0) \neq$ it maps a neighbourhood of z_0 conformally and topologically onto a region.

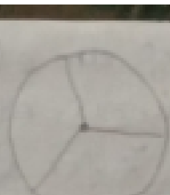
Proof:

Local correspondence.

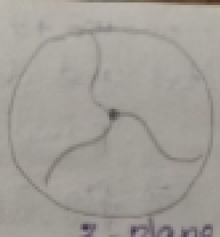


z-plane fig(1) w-plane fig

w-plane fig(3)



ζ-plane fig(4)



z-plane fig(5).

(87) $\Rightarrow$ If the local mapping is one to one (thm 11) can be hold only with $n=1$ / and

hence $f'(z_0)$ must be different from zero.

$\Rightarrow$ for $n>1$ the local correspondence can still be described in very precise terms

$\Rightarrow$ under the assumption of (thm 11).

we can write

$$f(z) - w_0 = (z-z_0)^n g(z)$$

where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.

$\Rightarrow$ choose $\varepsilon > 0$ so that $|g(z) - g(z_0)| < |g(z_0)|$ for $|z-z_0| < \varepsilon$.

$\Rightarrow$ In this neighbourhood it is possible to define a single valued analytic branch of $\sqrt[n]{g(z)}$

which we denote by $h(z)$

$\Rightarrow$ we have thus

$$f(z) - w_0 = f(\zeta)^n$$

$$\zeta(z) = (z-z_0) h(z)$$

$\Rightarrow$ Since $\zeta'(z_0) = h(z_0) \neq 0$.

the mapping $z = \zeta(z)$ is topological in a neighbourhood of z_0.

$\Rightarrow$ on the otherhand the mapping

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$w = w_0 + z^n$ is of an elementary character and determinate n equally.

Spaced value z for each value w.

88 $\Rightarrow$ fig 3, 4, 5 show the inverse image of a small disk and the n-arcs which are mapped on to the positive radius.

Theorem: 12 (The maximum principle).

If $f(z)$ is analytic and non constant in a region Ω then its absolute value $|f(z)|$ has no maximum in Ω.

Proof:

$\Rightarrow$ If $w_0 = f(z_0)$ is any value taken in Ω there exists a neighbourhood

$|w - w_0| < \varepsilon$

contained in the image of Ω.

$\Rightarrow$ In this neighbourhood there are point of modulus $> |w_0|$ and hence $|f(z_0)|$ is not maximum of $|f(z)|$.

Theorem: 12'. (Schwarz lemma).

If $f(z)$ is defined and continuous on a closed bounded set E and analytic on the interior of E, then the maximum of $|f(z)|$ on E is assumed on the boundary of E.

Proof:

$\Rightarrow$ Since E is compact.

$|f(z)|$ has a maximum on E.

$\Rightarrow$ Suppose that it is assumed at z_0

If z_0 is on the boundary there is nothing to prove,

$\Rightarrow$ If z_0 is an interior point then $|f(z_0)|$ is also the maximum of $|f(z)|$ in a disk

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$|z - z_0| < \delta$ contained in E

⇒ But this is not possible unless $f(z)$ is constant in the component of the interior of E which contains z_0.

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⇒ It follows by continuity that $|f(z)|$ is equal to its maximum on the whole boundary of that component.

⇒ Thus boundary is not empty and it is contained in the boundary of E.

⇒ Thus the maximum is always assumed at a boundary point.

⇒ The maximum principle can also be proved analytically as a consequence of cauchy integral formula.

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi) d\xi}{\xi - z} \rightarrow ①$$

We can write $\xi = z_0 + re^{i\theta}$,

$$d\xi = i r e^{i\theta} d\theta$$

on γ. and obtain for $z = z_0$

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta \rightarrow ②$$

⇒ This formula shows that the value of an analytic function at the centre of a circle is equal to the arithmetic mean of values on the circle

⇒ Subject to the condition that the closed disk $|z - z_0| \le r$ is contained in the region of analyticity.

$$|f(z_0)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta \rightarrow ③$$

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⇒ Suppose that $|f(z_0)|$ were a maximum

90 ⇒ Then we would have $|f(z_0+re^{i\theta})| \le |f(z_0)|$

⇒ and if the strict inequality held for a single value of θ it would hold by continuity on a whole arc.

⇒ But then the mean value of $|f(z_0+re^{i\theta})|$ would be strictly less than $|f(z_0)|$ and ③ would lead to the contradiction

$$|f(z_0)| < |f(z_0)|.$$

⇒ Thus $|f(z)|$ must be constantly equal to $|f(z_0)|$ on all sufficiently small circles $|z-z_0| = r$

and hence in a neighbourhood of z_0.

⇒ It follows easily that $f(z)$ must reduce to a constant.

⇒ This reasoning provides a second proof of the maximum principle.

⇒ consider now the case of a function $f(z)$ which is analytic in the open disk

$$|z| < R$$

and continuous on the closed disk

$$|z| \le R.$$

⇒ If it is known that $|f(z)| \le M$ on $|z| = R$ then $|f(z)| \le M$

in the whole disk

⇒ The equality can hold only if $f(z)$ is a constant of absolute value M.

⇒ Therefore if it is known that $f(z)$ takes some value of modulus.

Theorem: 13

If $f(z)$ is analytic for $|z|<1$ and satisfies the conditions $|f(z)| \le 1$, $f(0) = 0$, then

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$|f(z)| \le |z|$ and $|f'(0)| \le 1$. If ...

~~$|f(z)| \le |z|$ and $|f'(0)| \le 1$.~~

$|f(z)| = |z|$ for some $z \neq 0$, or if $|f'(0)| = 1$.

(91) then $f(z) = cz$ with a constant c of absolute value 1.

proof:

we apply the maximum principle to the function $f_1(z)$ which is equal to $f(z)/z$. for $z \neq 0$ and to $f'(0)$ for $z = 0$.

$\Rightarrow$ on the circle $|z| = r < 1$ it is of absolute value $\le 1/r$.

and hence

$|f_1(z)| \le 1/r$ for $|z| \le r$.

$\Rightarrow$ Letting r tend to 1

we find that $|f_1(z)| \le 1$ $\forall z$.

and this is the assertion of the theorem

$\Rightarrow$ If the equality holds at a single point it means that $|f_1(z)|$ attains its maximum.

and hence that $f_1(z)$ must reduce to constant.

$\Rightarrow$ If $f(z)$ is known to satisfy the conditions of the theorem in a disk of radius R, the original form of the theorem can be applied to the function $f(Rz)$.

$\Rightarrow$ As a result we obtain

$|f(Rz)| \le |z|$.

which can be rewritten as $|f(z)| \le |z|$

$\Rightarrow$ Similarly if the upper bound of the modulus is M instead of 1.

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we apply the theorem to $f(z)/M$ or in the more general case to $f(Rz)/M$.

=> The resulting inequality is

92 $|f(z)| \le M|z|/R$.

=> Still more generally we may replace the condition $f(z_0) = w_0$.

where $|z_0| < R$ and $|w_0| < M$.

=> let $\zeta = Tz$ be a linear transformation which maps $|z| < R$ onto $|\zeta| < 1$.

with z_0 going into the origin and let Sw be a linear transformation which $Sw_0 =$ which maps $|w| < M$ onto $|Sw| < 1$.

=> It is clear that the function $Sf(T^{-1}\zeta)$.

UNIT-IV

The calculus of Residues :-

Defn : Residue :

The Residue of $f(z)$ at an isolated singularity a is the unique complex number R which makes $f(z) - R/(z-a)$ the derivative of a single - valued analytic function in an annulus $0 < |z-a| < \delta$

Cauchy's Integral formula using for residue Theorem

If $f(z)$ is analytic in a region Ω then $n(\gamma, a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z) \, dz}{z-a}$ for a cycle γ which is homologous to zero

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State and prove Residue theorem:

(93) Let $f(z)$ be analytic except for isolated singularities a_j in a region Ω. then $\frac{1}{2\pi i}\int f(z)dz = \sum_j n(\gamma, a_j)\ Res_{z=a_j} f(z)$ for any circle γ which is homologous to zero in Ω and does not pass through any of the point a_j.

proof:

Let $f(z)$ be analytic except for isolated singularities a_j in a region Ω.

In the case each $n(\gamma, a_j)$ is either 0 or 1.

Then we have simply

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_j Res_{z=a_j} f(z).$$

where the sum of extended overall singularities enclosed by γ.

we need only look at the expansions

$f(z) = B_n(z-a)^{-n} + \dots + B_1(z-a)^{-1} + \phi(z)$

to recognize that the residue equals the co-efficient B_1.

$\Rightarrow$ when the term $B_1(z-a)^{-1}$ is omitted, the remainder is evidently a derivative.

Since the principal part at a pole is always either given or can be easily found, we have thus a very simple method for finding the residues

Defn:- Bound of the region Ω:

A cycle γ is said to bounded the region Ω if and only if $n(\gamma, a)$ is defined and equal to 1 for all points at Ω and either undefined or equal to zero for all points a not in Ω.

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If γ bounds Ω and $f(z)$ is analytic on the set $\Omega + \gamma$, then

$$\int_\gamma f(z)\,dz = 0 \quad \text{and}$$

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z} \quad \text{for all } z \in \Omega$$

If $f(z)$ is analytic on $\Omega + \gamma$ except for isolated singularities in Ω, then

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_j \operatorname{Res}_{z=a_j} f(z).$$

where the sum ranges over the singularities $a_j \in \Omega$.

The argument principle:

The number of zeros of an analytic function for a zero of order h we can write

$f(z) = (z-a)^h f_h(z)$, with $f_h(a) \neq 0$, and obtain $f'(z) = h(z-a)^{h-1} f_h(z) + (z-a)^h f_h'(z)$

Consequently

$$f'(z)/f(z) = h/(z-a) + f_h'(z)/f_h(z)$$

and we see that f'/f has a simple pole with the residue h.

Theorem: 18 argument principle.

~~State and prove Rouche's theorem:~~

Statement:

If $f(z)$ is meromorphic in Ω with

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the zeros a_j and the poles b_k, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k),$$

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for every cycle γ which is homologous to zero in Ω and does not pass through of any of the zeros or poles.

proof:

Let $f(z)$ is merimorphic in Ω with the zeros a_j and the poles b_k.

To prove that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k) \quad \text{①}$$

It is understood the multiple zeros and poles have to be repeated as many times as their order indicates.

This is the eqn ① can be are finite

The name refers to the interpretation of the left hand member of ① as $n(\Gamma, 0)$.

where Γ is the image cycle of γ.

If Γ lies in a disk which does not contain the origin then $n(\Gamma, 0) = 0$.

This observation is the basis for the following corollary, known as Rouche's theorem.

1) State and prove Rouche's theorem.

corollary:

Let γ be a homologous to zero in Ω and such that $n(\gamma, z)$ is either 0 or 1 for any point z not on γ. Suppose that $f(z)$ and $g(z)$ are

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analytic in Ω and satisfy the inequality the inequality $|f(z)-g(z)| < |f(z)|$ on γ.

96 Then $f(z)$ and $g(z)$ have the same number of zeros enclosed by γ.

Proof:

The assumption implies that $f(z)$ and $g(z)$ are zero free on γ.

Moreover, they satisfy the inequality $\left| \frac{g(z)}{f(z)} - 1 \right| < 1$ on γ.

The values of $F(z) = g(z)/f(z)$ on γ. or thus contained in the open disk of centre 1 and radius 1.

we have thus

$$n(\Gamma, 0) = 0.$$

A function $f(z)$ in the disk $|z| \leq R$. using Taylor's theorem we can write

$$f(z) = P_{n-1}(z) + z^n f_n(z)$$

If $g(z)$ is analytic in Ω, then $g(z) \frac{f'(z)}{f(z)}$ has the residue $h_g(a)$ at a zero a of order h and the residue $-hg(a)$ at a pole.

$$\frac{1}{2\pi i} \int_\gamma g(z) \frac{f'(z)}{f(z)} dz = \sum_j n(\gamma, a_j) g(a_j)$$

$$- \sum_k n(\gamma, b_k) g(b_k) \rightarrow (1).$$

This result is important for the study of the inverse function.

Then eqn $f(z) = w$, $|w - w_0| < \delta$ has n roots

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in the disk $|z - z_0| < \epsilon$.

If we apply ① with $g(z) = z$, we obtain.

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$$\sum_{j=1}^{n} z_j(\omega) = \frac{1}{2\pi i} \int_{|z-z_0|=\epsilon} \frac{f'(z)}{f(z)-\omega} z\, dz \quad \text{—②}$$

for $n=1$ the inverse function $f^{-1}(\omega)$ can thus be represented explicity by

$$f^{-1}(\omega) = \frac{1}{2\pi i} \int_{|z-z_0|=\epsilon} \frac{f'(z)}{f(z)-\omega} z\, dz.$$

If ① is applied with $g(z) = z^m$

eqn ② is replaced by

$$\sum_{j=1}^{n} z_j(\omega)^m = \frac{1}{2\pi i} \int_{|z-z_0|=\epsilon} \frac{f'(z)}{f(z)-\omega} z^m\, dz.$$

The right-hand number represents an analytic function of ω for $|\omega - \omega_0| < \delta$.

we find that the $z_j(\omega)$ are the roots of a polynomial eqn

$$z^n + a_1(\omega) z^{n-1} + \dots + a_{n-1}(\omega) z + a_n(\omega) = 0.$$

whose co-efficients are analytic functions of ω in $|\omega - \omega_0| < \delta$.

Evaluation of definite integral:

There are the five complex integral of evaluate of definite integrals.

1. All integrals of the form.

2. An integral of the form.

3. The some method can be applied to an integral of the form.

4. The next category of integrals have the f

5. As a final example we compute the special integral.

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1. Integral of the form:

Let the evaluation for definite integral for all the integral of the form.

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$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta \rightarrow ①$$

where by integration is Rational function of R cos θ and sin θ can be evaluated means of residues.

⇒ It is very natural to make the substitution $z = e^{i\theta}$

which immediately transforms of the eqn of ① is line integral.

$$-i \int_{|z|=1} R\left[\tfrac{1}{2}(z+\tfrac{1}{z}), \tfrac{1}{2i}(z-\tfrac{1}{z})\right] \frac{dz}{z}$$

the residues which corresponds to the whole of the integrate inside the unique circle.

for ex:

Let the compute $\int_0^{\pi} \frac{d\theta}{a+\cos\theta}$, $a>1$ is integral is not extended over $(0, 2\pi)$.

But since cos θ takes the same values in the intervals $(0,\pi)$ and $(\pi, 2\pi)$.

It is clear that the integral form 0 to π.

1/2 one half of the integral form 0 to 2π. This is the integral for a tab.

⇒ Taking $-i \int_{|z|=1} \frac{dz}{z^2+2az+1}$

the dominator can be factor in to $(z-\alpha)$ $(z-\beta)$ with

$\alpha = -a + \sqrt{a^2-1}$, $\beta = -a - \sqrt{a^2-1}$.

i.e) $|\alpha|<1$, $|\beta|>1$ and the residues at α $1/\alpha-\beta$.

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The value of the integral ~~of~~ forms to be
$\pi / \sqrt{a^2-1}$.

(99)

2. An integral of the form:

Evaluate An — Let the evaluation for definite integral for ~~all~~ ~~the~~ integral of the form

$$\int_{-\infty}^{\infty} R(x)\,dx \rightarrow ②$$

eqn ② is called the convergence.

If and only if in the rational functions $R(x)$.

the degree of the denominator is atleast two ~~units~~ limits higher than degree of the numerator and if no pole lies on a real axis.

The standard procedure is to integrate the Complex function $R(z)$ over a closed curve consisting of a line segment $(-\rho,\rho)$.

In the upper half plane.

$\Rightarrow$ If ρ is large enough this curve enclosures all poles in the upper half plane. to corresponding Integrals equal to $2\pi i$ times the same of the residues in the upper half plane.

$\Rightarrow$ As $\rho \to \infty$ of obviously to the integral over semi circle ~~dense~~ tends to zero.

and we obtain

$$\int_{-\infty}^{\infty} R(x)\,dx = 2\pi i \sum_{y>0} \text{Res } R(z).$$

3. The same method can be applied to an integral of the form

Evaluate $\int_{-\infty}^{\infty} R(x)e^{ix}\,dx \rightarrow ③$

whose real and imaginary parts determine the important integrals

$$\int_{-\infty}^{\infty} R(x)\cos x\,dx, \quad \int_{-\infty}^{\infty} R(x)\sin x\,dx \rightarrow ④$$

Since $|e^{iz}| = e^{-y}$ is bounded in the upper half pla

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the rational function $R(x)$ has a zero of atleast order two at infinity.

We obtain

$$\int_{-\infty}^{\infty} R(x) e^{ix}\, dx = 2\pi i \sum_{y>0} \operatorname{Res} R(z) e^{iz}$$

the integral

$$\int_{-\rho}^{\rho} R(x) e^{ix}\, dx$$

over a symmetric interval has the desired limit for $\rho \to \infty$.

In reality we are of course required to prove that

$$\int_{-X_1}^{X_2} R(x) e^{ix}\, dx$$

has a limit when X_1 and X_2 tend independently to ∞.

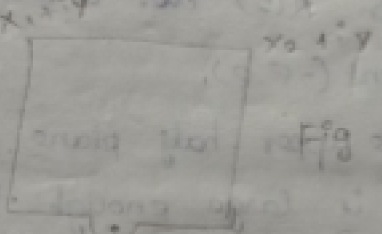

Fig 4.12.

For the proof we integrate over the perimeter of a rectangle with the vertices X_2, $X_2 + iY$, $-X_1 - iY$, $-X_1$ where $Y > 0$.

As soon as X_1, X_2 and Y are sufficiently large this rectangle contains all the poles in the upper half plane.

Under the hypothesis $|z(R(z))|$ is bounded.

Hence the integral over the Right vertical side is except for a constant factor, less than

$$\int_0^Y e^{-y} \frac{dy}{|z|} < \frac{1}{X_2} \int_0^Y e^{-y}\, dy$$

The last integral can be evaluated explicitly and is found to be < 1.

Hence the integral over the right vertical side is less than a constant times $1/X_2$. and a corresponding result is found for the left vertical side.

The integral over the ~~right vertical~~

horizontal side is evidently less than $e^{-Y}(X_1+X_2)/Y$ multiplied with a constant.

For fixed X_1, X_2 it tends to 0 as $Y \to \infty$

101 and we conclude that

$$\left| \int_{-X_1}^{X_2} R(x) e^{ix} dx - 2\pi i \sum_{y>0} \text{Res}\, R(z) e^{iz} \right| < A\left(\frac{1}{X_1} + \frac{1}{X_2}\right)$$

where A denotes a constant. This inequality proves that

$$\int_{-\infty}^{\infty} R(x) e^{ix} dx = 2\pi i \sum_{y>0} \text{Res}\, R(z) e^{iz}$$

under the sole condition that $R(\infty) = 0$.

It is clear that we are led to the result.

$$\boxed{\lim_{\delta \to 0} \int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} R(x) e^{ix} dx = 2\pi i \left[\sum_{y>0} \text{Res}\, R(z) e^{iz} + \tfrac{1}{2} B \right]}$$

The limit on the left is called the cauchy principle value of the integral.

In the general case where several poles lie on the real axis we obtain.

$$\text{pr. v.} \int_{-\infty}^{\infty} R(x) e^{ix} dx = 2\pi i \sum_{y>0} \text{Res}\, R(z) e^{iz} + \pi i \sum_{y=0} \text{Res}\, R(z) e^{iz}$$

where the notations are self explanatory.

Assume that $R(\infty) = 0$.

As the simplest example we have

$$\text{Pr.v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \pi i.$$

Since the integrand in even we find

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

We remark that integrals containing a factor $\cos^n x$ or $\sin^n x$ can be evaluated by the same techniques.

these factors can be written as linear combinati of terms $\cos mx$ and $\sin mx$ and the corresponding Integrals can be reduced to the form ③ by a change variable.

$$\int_{-\infty}^{\infty} R(x)\, e^{imx}\, dx = \frac{1}{m} \int_{-\infty}^{\infty} R\left(\frac{x}{m}\right) e^{ix}\, dx.$$

102 4. The next category of integrals have the form:

$$\int_0^{\infty} x^{\alpha} R(x)\, dx.$$

=> where the exponent α is a real and may be supposed to lie in the interval $0<\alpha<1$.

=> for convergence $R(x)$ must have a zero of atleast order two at ∞ and atmost simple pole at the origin.

The new fig is fact that $R(z)\, z^{\alpha}$ is non single value this is the circumstances

This match it possible to find the integral from ~~$-\infty$~~ 0 to ∞.

The simple to the start with the subtution $x = t^2$ which transforms the integral in to

~~for~~

for the fn $z^{2\alpha}$ we make choose the strange choose argument lies between $-\pi\alpha$ & $3\pi\alpha$.

=) we can apply the residue theorem thm to the fn $z^{2\alpha+1} + R(z^{\alpha})$.

hence the residue theorem using the value of integral

$$\int_{-\infty}^{\infty} z^{2\alpha+1} R(z^2)\, dz = \int_0^{\infty} (z^{2\alpha+1} + (-z)^{2\alpha+1}\, R(z^2)\, dz$$

However $(-z)^{2\alpha} = e^{2\pi i\alpha} z^{2\alpha}$.

and the integral equals

$$(1-e^{2\pi i\alpha}) \int_0^{\infty} z^{2\alpha+1}\, R(z^2)\, dz.$$

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Consider the function Evaluate $\int_0^{\pi} \log \sin\theta \, d\theta$ where

103 $1-e^{2iz} = -2ie^{iz}\sin z$.

$\Rightarrow$ for the representation $1-e^{2iz} = 1-e^{-2y}(\cos 2x + i\sin 2x)$

$\Rightarrow$ This fn is real and negative only for $x = n\pi$, $y \leq 0$.

$\Rightarrow$ In the region obtained by omitting this half lines the principle branch of $\log(1-e^{2iz})$ is single value and analytic.

You can apply for the cauchy theorem to the rectangle where vertices are $0, \pi, \pi+iy$ and iy.

The points are 0 & π have to be avoided and so to these following the small circular co-ordinates of radius δ.

$\Rightarrow$ The integrals over the upper horizontal tends to zero as $y \to \infty$.

$\Rightarrow$ The integral over the co-ordinates can be approaches $\delta \to 0$.

$\Rightarrow$ The imaginary part of logarithms is bounded we need only consider the next part.

$\Rightarrow$ for fact that $|1-e^{2iz}| / |z| \to 2$ for $z \to 0$, we see that

$\log|1-e^{2iz}|$ becomes infinite like $\log\delta$, and since $\delta\log\delta \to 0$

The integral over the quadrants have the origin will tend to zero.

$= 1-e^{-2y}(\cos 2x + i\sin 2x)$.

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The same proof apply the near version π and we obtain

$$\int_0^{\pi} \log(-2ie^{ix} \sin x)\, dx = 0.$$

If we choose $\log e^{ix} = ix$,

the imaginary part lies between 0 and

⇒ therefore in order to obtain the principle branch with imaginary part and $-\pi$ and π.

⇒ we must choose $\log(-i) = -\pi i/2$

the eqn can now be written in the form

$$\pi \log 2 - (\pi^2/2)i + \int_0^{\pi} \log \sin x\, dx + (\pi^2/2)i = 0.$$

and we find

$$\int_0^{\pi} \log \sin x\, dx = -\pi \log 2.$$

Bergman's kernal formula:

The complex integration can sometime we used to evaluate a integrals in the the using the show that if $f(z)$ is analytic and bounded for

$|z|<1$ and if $|\xi|<1$ then

$$f(\xi) = \frac{1}{\pi} \int_{|z|<1} \frac{f(z)\, dx\, dy}{(1-\bar{z}\xi)^2}.$$

UNIT - V

power series expansions.

Theorem : 1
Weierstrass theorem :

~~Suppose~~ If that $f_n(z)$ is analytic in a region Ω_n, and that the sequence $\{f_n(z)\}$ converge to a limit functions $f(z)$ in a region Ω, uniformly on for every compact subset Ω. Then $f(z)$ is analytic in Ω. Moreover, $f_n'(z)$ converges uniformly to $f'(z)$ on every compact subset of Ω.

proof:

Let $f_n(z)$ is analytic in a region Ω_n

Let $|z-a| \leq r$. ($\because$ by using morera's theorem) be a closed disk contained in Ω.

Let us assumption is implies that the disk lies in Ω_n for all $n >$ certain n_0.

$\Rightarrow$ Let γ is any closed curve containing in $|z-a| < r$.

we have

$$\int_\gamma f_n(z)\, dz = 0.$$

For $n > n_0$ (by using the cauchy's theorem)

Because of the uniform convergence on γ we obtain

$$\int_\gamma f(z) dz = 0.$$

$$= \lim_{n \to \infty} \int_\gamma f_n(z) dz = 0.$$

the morera's thm is follows that $f(z)$ is analytic in $|z-a| < r$

consequently $f(z)$ is analytic in the whole region Ω.

$\Rightarrow$ An alternative is basic on the integral

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$f_n(z) = \frac{1}{2\pi i} \int \frac{f_n(\xi)\,d\xi}{\xi - z}$

106 where c is the circle

$|\xi - a| = r$.

Let $n \to \infty$ we obtain by uniform convergence $f(z) = \frac{1}{2\pi i} \int_c \frac{f(\xi)\,d\xi}{\xi - z} \to ①$

This eqn ① is shows that $f(z)$ is analytic in the disk

$\Rightarrow$ Starting from the formula

$f_n'(z) = \frac{1}{2\pi i} \int_c \frac{f_n(\xi)}{(\xi - z)^2}\,d\xi$.

The same reasoning its $\lim_{n\to\infty} f_n'(z) =$

$\frac{1}{2\pi i} \int_c \frac{f(\xi)}{(\xi - z)^2}\,d\xi$

$= f'(z). \longrightarrow ②$

The eqn ② is convergence is uniform for $|z - a| \leq \rho < r$.

$\Rightarrow$ Therefore any compact subset of Ω can be covered by closed disk

Therefore the convergence is uniform on every compact subset.

(i.e) If a series with analytic terms $f(z) = f_1(z) + f_2(z) + \dots + f_n(z) + \dots$ converges uniformly on a every compact subset of a region Ω

Then the sum $f(z)$ is analytic in Ω and the series can be differentiates by term)

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Theorem: 2

107 If the function $f_n(z)$ are analytic and $\neq 0$ in a region Ω and if $f_n(z)$ converges to $f(z)$, uniformly on a every compact subset Ω, then $f(z)$ is either identically zero or never equal to 0 in Ω.

proof:

Let the function $f_n(z)$ are analytic.

Suppose that not identically zero.

$\Rightarrow$ The zeros of $f(z)$ are in any case isolated

$\Rightarrow$ For any point $z_0 \in \Omega$.

There is therefore a number $r>0$ such that

$f(z)$ is defined and $\neq 0$ for $0<|z-z_0|\leq r$

$\Rightarrow$ In particular $|f(z)|$ has a positive minimum on the circle $|z-z_0|=r$.

which we denote by c.

It is follows that $\frac{1}{f_n(z)}$ converges uniformly to $\frac{1}{f_n(z)}$ on c.

Since it is also true that

$f_n'(z) \to f'(z)$

uniformly on c.

$\Rightarrow$ we may conclude that the $\lim_{n\to\infty}$

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_c \frac{f_n'(z)}{f_n(z)}\, dz.$$

$$= \frac{1}{2\pi i} \int_c \frac{f'(z)}{f(z)}\, dz.$$

But the integrals on the left are all zero for they give the number of roots of the any $f_n(z)=0$. inside of c.

$\Rightarrow$ The integral on the right is therefore

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and consequently $f(z_0) \neq 0$. by the same integration of the integral.

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Since z_0 was arbitary.

The taylor's series:

If $f(z)$ is analytic in a region Ω containing z_0 we can write $f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \dots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + f_{n+1}(z)(z-z_0)^{n+1}$

proof:

If $f_{n+1}(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)\,d\xi}{(\xi-\xi_0)^{n+1}(\xi - z)}$.

in the last formula C is any circle. $|z_0 - z_0| = \rho$.

Such that the closed disk $|z - z_0| \leq \rho$ is contained in Ω.

proof:

$\Rightarrow$ If M denotes the max $|f(z)|$ on c.

we obtain at one the estimate.

$$|f_{n+1}(z)(z-z_0)^{n+1}| \leq \frac{M|z-z_0|^{n+1}}{\rho^n(\rho - |z-z_0|)}$$

the term is uniformly 0.

In every disk $|z - z_0| \leq r < \rho$.

Thus ρ is the closed to the shortest distance from z_0.

To the boundary of Ω.

$$\int_{C_1} f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \dots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + f_{n+1}(z)(z-z_0)^{n+1}$$

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canonical products:

A function is which is analytic in the

109 whole plane is said to be entire (or) Integral.

The simplest entire functions which are not polynomials are e^z, $\sin z$, and $\cos z$.

If $g(z)$ is an entire function, then

$\Rightarrow f(z) = e^{g(z)}$ is entire and $\neq 0$.

Conversely,

$\Rightarrow$ if $f(z)$ is any entire function which is never zero, let us show that $f(z)$ is of the form $e^{g(z)}$

$\Rightarrow$ To this end we observe that the function $f'(z)/f(z)$, being analytic in the whole plane, is the derivative of an entire function $g(z)$.

$\Rightarrow$ If $f(z)$ has m zeros at the origin (m may be zero), and denote the other zeros by $a_1, a_2 \ldots a_N$ multiple zeros being Repeated.

It is then plain that

$\Rightarrow$ we can write

$$\Rightarrow f(z) = z^m e^{g(z)} \prod_1^{\infty} \left(1 - \frac{z}{a_n}\right).$$

$\Rightarrow$ The obvious generalization would be

$$f(z) = z^m e^{g(z)} \cdot \prod_1^{\infty} \left(1 - \frac{z}{a_n}\right) \longrightarrow ①$$

$\Rightarrow$ The product in ① converges absolutely iff

$\Rightarrow \sum_1^{\infty} 1/|a_n|$ is convergent.

$\Rightarrow$ The convergence is also uniform in every closed disk $|z| \leq R$.

$\Rightarrow$ An arbitary sequence of complex numbers $a_n \neq 0$ with $\lim_{n\to\infty} a_n = \infty$, and prove the existence of polynomials $P_n(z)$ such that

$$\prod_1^{\infty} \left(1 - \frac{z}{a_n}\right) e^{P_n(z)} \longrightarrow ②$$

converges to an entire function.

The product converge together with the series with the general term.

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$$r_n(z) = \log\left(1 - \frac{z}{a_n}\right) + p_n(z) \text{ [illegible]}$$

⇒ where the branch of the logarithm shall be chosen so that the imaginary part of $r_n(z)$ lies between $-\pi$ and π (inclusive)

⇒ for a given R we consider only the terms with $|a_n| > R$.

In the disk $|z| \leq R$ the principal branch of $\log(1 - z/a_n)$ can be developed in a taylor series

$$\log\left(1 - \frac{z}{a_n}\right) = -\frac{z}{a_n} - \frac{1}{2}\left(\frac{z}{a_n}\right)^2 - \frac{1}{3}\left(\frac{z}{a_n}\right)^3 \cdots$$

the signs and choose $p_n(z)$ as a partial sum

$$p_n(z) = \frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}$$

then $r_n(z)$ has the representation

$$r_n(z) = -\frac{1}{m_n+1}\left(\frac{z}{a_n}\right)^{m_n+1} - \frac{1}{m_n+2}\left(\frac{z}{a_n}\right)^{m_n+2} \cdots$$

and we obtain easily the estimate

$$|r_n(z)| \leq \frac{1}{m_n+1}\left(\frac{R}{|a_n|}\right)^{m_n+1}\left(1 - \frac{R}{|a_n|}\right)^{-1} \rightarrow ③.$$

Suppose now that the series

$$\sum_{n=1}^{\infty} \frac{1}{m_n+1}\left(\frac{R}{|a_n|}\right)^{m_n+1} \rightarrow ④.$$

Converges.

⇒ the series $\sum r_n(z)$ is absolutely and uniformly convergent for $|z| \leq R$.

⇒ and thus the product ② represents an analytic function in $|z| < R$.

⇒ for the sake of the reasoning we had to exclude the values $|a_n| \leq R$.

⇒ but it is clear that the uniform convergence of ② is not affected when the corresponding factors are again taken into account.

It remains only to show that the series

④ can be made convergent for all R.

we take $m_n = n$ it is clear that ④ has a

111 majorant geometric series with ratio < 1 for any fixed value of R.

Theorem: 7

There exists an entire function with arbitrarily prescribed zeros an provided that, in the case of infinitely many zeros, $a_n \to \infty$.

Every entire function with these and no other zeros can be written in the form.

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \dots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)} \to ⑤.$$

corollary:

Every function which is meromorphic in the whole plane is the quotient of two entire function

Proof:

By using above theorem.

The product $F(z) g(z)$ is then an entire function $f(z)$, and we obtain

$F(z) = f(z)/g(z)$.

The proceeding like this the product

$$\prod_{1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \dots + \frac{1}{h}\left(z/a_n\right)^h} \to ⑥$$

converges and represents an entire function provided that the series

$$\sum_{n=1}^{\infty} (R/|a_n|)^{h+1}/(h+1) \text{ converges for all } R.$$

that is to say provided that

$$\sum 1/|a_n|^{h+1} < \infty.$$

Assume that h is the smallest integer for which this series converges.

Then eqn ⑥ is then called the canonical product the eqn ⑥ is associated with the sequence

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$\{a_n\}$, and h is the genus of the canonical product.

$\Rightarrow$ an entire function of genus zero is of the form

$$Cz^m \prod_1^\infty \left(1 - \frac{z}{a_n}\right)$$

with $\sum 1/|a_n| < \infty$.

$\Rightarrow$ The canonical representation of an entire function of genus 1 is either of the form

$$Cz^m e^{\alpha z} \prod_1^\infty \left(1 - \frac{z}{a_n}\right) e^{z/a_n}$$

with $\sum_1 1/|a_n|^2 < \infty$, $\sum_1 1/|a_n| = \infty$, or of the form

$$Cz^m e^{\alpha z} \prod_1^\infty \left(1 - \frac{z}{a_n}\right).$$

with $\sum 1/|a_n| < \infty$, $\alpha \neq 0$.

$\Rightarrow$ As an application we consider the product representation of $\sin \pi z$.

$\Rightarrow$ The zeros are the integers $z = \pm n$.

$\Rightarrow$ Since $\sum 1/n$ diverges and $\sum_1 1/n^2$ converges, we must take $h = 1$ and obtain a representation of the form

$$\sin \pi z = z e^{g(z)} \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n}$$

$\Rightarrow$ In order to determine $g(z)$ we form the logarithmic derivative on both sides.

$\Rightarrow$ we find

$$\pi \cot \pi z = 1/z + g'(z) + \sum_{n \neq 0} \left(\frac{1}{z-n} + 1/n\right)$$

where the procedure is easy to justify by uniform convergence on any compact set which does not contain the points $z = n$.

$\Rightarrow$ comparison with the previous formula

$\Rightarrow$ we conclude that $g'(z) = 0$.

$\Rightarrow$ Hence $g(z)$ is a constant, and since

$\lim_{z \to 0}$

$\lim_{z\to 0} \sin\pi z / z = \pi$ we in

we must have $e^{g(z)} = \pi$.

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and thus

$$\sin\pi z = \pi z \prod_{1}^{\infty}\left(1-\frac{z}{n}\right)e^{z/n} \rightarrow ①$$

the factors corresponding to n and $-n$ can be bracketed together and we obtain the simple form.

$$\sin\pi z = \pi z \prod_{1}^{\infty}\left(1-\frac{z^2}{n^2}\right) \rightarrow ②$$

It follows from ① that $\sin\pi z$ is an entire function of genus 1.

The Laurent series:

A series of the form

$$b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_n z^{-n} + \dots \rightarrow ①$$

can be considered as an ordinary power series in the variable $1/z$. It will therefore converge outside of some circle $|z| = R$, except in the extreme case $R = \infty$;

Proof

The convergence is uniform in every region $|z| \geq \rho > R \rightarrow ①$

and hence the series ① is combined with an ordinary power series, we get a more general series of the the form.

$$\sum_{n=-\infty}^{\infty} a_n z^n \rightarrow ②$$

It will be termed convergent only if the parts consisting of nonnegative powers and negative powers are separately convergent.

Since the first part converges in a disk $|z| < R_2$ and the second series in a region $|z| > R_1$, there is common region of convergence only if $R_1 < R_2$, and ② represents an analytic function in the annulus

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$R_1 < |z| < R_2$.

Conversely,

we may start from an analytic function $f(z)$ whose region of definition contains an annulus $R_1 < |z| < R_2$.

Or more generally an annulus $R_1 < |z-a| < R$

we shall show that such that a function can always be developed in a general power series of the form

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z-a)^n$$

The proof is extremely simple

All we have to show that $f(z)$ can be written as a sum

$\Rightarrow f_1(z) + f_2(z)$

where $f_1(z)$ is analytic for $|z-a| < R_2$.

and $f_2(z)$ is analytic for $|z-a| > R_1$.

with a removable singularity at ∞

under these circumstances $f_1(z)$ can be developed in non-negative powers of $z-a$, and

$f_2(z)$ can be developed in non-negative powers of $1/(z-a)$.

To find the representation

$$f(z) = f_1(z) + f_2(z)$$

defined $f_1(z)$ by

$$f_1(z) = \frac{1}{2\pi i} \int_{|\xi-a|=r} \frac{f(\xi)\,d\xi}{\xi - z}$$

for $|z-a| < r < R_2$ and

$f_2(z)$ by

$$f_2(z) = -\frac{1}{2\pi i} \int_{|\xi-a|=r} \frac{f(\xi)\,d\xi}{\xi - z}$$

for $R_1 < r < |z-a|$.

In both integrals the value of r is

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Irrelevent as long as the inequality is fulfilled.

for it is an immediately consequence of cauchy theorem that the value of the integral does not chang with r provided that the circle does not pass over the point z.

115 $\Rightarrow$ for this reason $f_1(z)$ and $f_2(z)$ are uniquely defined and represent analytic functions in $|z-a|<$ and $|z-a|>R_1$ respectively.

moreover, by cauchy's integral thm

$$f(z) = f_1(z) + f_2(z)$$

the taylor development of $f_1(z)$ is

$$f_1(z) = \sum_{n=0}^{\infty} A_n (z-a)^n \text{ with}$$

$$A_n = \frac{1}{2\pi i}\int_{|\zeta - a| = r} \frac{f(\zeta)\,d\zeta}{(\zeta-a)^{n+1}} \longrightarrow (3)$$

$\Rightarrow$ In order to find the development of $f_2(z)$ perform the transformation $\zeta = a + 1/\zeta'$

$z = a + 1/z'$.

This transformation carries $|\zeta - a| = r$ in $|\zeta'| = 1/r$ with negative orientation any by sim calculations we obtain

$$\Rightarrow f_2(a + 1/z') = \frac{1}{2\pi i}\int_{|\zeta'| = 1/r} \frac{z'}{\zeta'} \frac{f(a + 1/\zeta')\,d\zeta'}{\zeta' - z'}$$

$$= \sum_{n=1}^{\infty} B_n z'^n$$

with

$$B_n = \frac{1}{2\pi i}\int_{|\zeta'| = 1/r} \frac{f(a + 1/\zeta')\,d\zeta'}{\zeta'^{\,n+1}}$$

$$= \frac{1}{2\pi i}\int_{|\zeta - a| = r} f(\zeta)(\zeta - a)^{n-1}\,d\zeta.$$

this formula shows that we can write

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z-a)^n.$$

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$\Rightarrow$ where all the co-efficients A_n are determined by ③.

$\Rightarrow$ Observe that the integral in ③ is independent of r as long as $R_1 < r < R_2$

116 If $R_1 = 0$ the point a is an isolated singularity and $A_{-1} = B_1$ is the residue at a.

for $f(z) - A_{-1}(z-a)^{-1}$ is the derivative of a single valued function in $0 < |z-a| < R_2$.

Jensen's formula:

If $f(z)$ is an analytic function then $\log|f(z)|$ harmonic except at the zeros of $f(z)$. Therefore if $f(z)$ is analytic and ~~free~~ from zeros in $|z| \le \rho$

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log|f(\rho e^{i\theta})|\,d\theta \longrightarrow ①$$

and $\log|f(z)|$ can be expressed by poisson's formula.

$\Rightarrow$ The eqn ① remains valid if $f(z)$ has zeros on the circle $|z| = \rho$

$\Rightarrow$ The simplest proof is by dividing $f(z)$ with one factor $z - \rho e^{i\theta_0}$ for each zero.

$\Rightarrow$ It is sufficient to show that

$$\log\rho = \frac{1}{2\pi}\int_0^{2\pi}\log|\rho e^{i\theta} - \rho e^{i\theta_0}|\,d\theta$$

(or)

$$\int_0^{2\pi}\log|e^{i\theta} - e^{i\theta_0}|\,d\theta = 0.$$

$\Rightarrow$ This integral is evidently independent of θ_0 and we have only to show that

$$\int_0^{2\pi}\log|1 - e^{i\theta}|\,d\theta = 0$$

But this is a consequence of the formula

$$\int_0^{\pi}\log\sin x\,dx = -\pi\log 2$$

proved in (thm 1a) we will now investigate what becomes of ① in the presence of zeros in [illegible]

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$|z| < \rho$

Denote these zeros by $a_1, a_2 \ldots a_n$ multiple zeros being repeated and assume first that $z = 0$ is not a zero.

Then the function

117 $$F(z) = f(z) \prod_{i=1}^{n} \frac{\rho^2 - \bar{a}_i z}{\rho(z - a_i)}$$

is free zeros in the disk,

and $|F(z)| = |f(z)|$ on $|z| = \rho$

consequently we obtain

$$\log |F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho e^{i\theta})| \, d\theta$$

and substituting the value of $F(0)$

$$\log |f(0)| = -\sum_{i=1}^{n} \log\left(\frac{\rho}{|a_i|}\right) + \frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho e^{i\theta})| \, d\theta \rightarrow (2)$$

This is known as Jensen's formula.

poisson ~~of~~ Its importance lies in the fact that it relates the modulus $|f(z)|$ on a circle to the moduli of the zeros

If $f(0) = 0$, the formula is somewhat more complicated.

writting $f(z) = cz^h + \ldots$

we apply (2) to $f(z)(\rho/z)^h$ and find that the left hand member must be replaced by

$\log |c| + h \log \rho$.

There is a similar generalization of poisson's formula.

All that is needed is to apply the ordin poisson formula to $\log |f(z)|$

we obtain

$$\log |f(z)| = -\sum_{i=1}^{n} \log \left| \frac{\rho^2 - \bar{a}_i z}{\rho(z - a_i)} \right| + \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re} \, \frac{\rho e^{i\theta} + z}{\rho e^{i\theta} - z} \log |f(\rho e^{i\theta})| \, d\theta \rightarrow (3)$$

provided that $f(z) \neq 0$, Eqn (3) is frequently referred as the poisson - Jensen formula.

Strictly speaking the proof is valid only if $f \neq$ on $|z| = \rho$.

118 But ⑳ shows that the integral on the right is a continuous function of ρ, and from there it is ~~easy~~ easy to infer that the integral in ⑤ is likewise continuous.

In general case ⑤ can therefore be derived by letting ρ approach a limit.

The Jensen and poisson - Jensen formulas have important applications in the theory of entire functions.

Thm: 3.

If $f(z)$ is analytic in the region Ω containing z_0, then the representation

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \dots + \frac{f^{(n)}(z_0)}{n!}(z - z_0) + \dots$$

is valid in the largest open disk of center z_0 contained in Ω.

proof:

⇒ The radius of convergence of the taylor series is thus atleast equal to the shortest distance from z_0 to the boundary of Ω.

⇒ It may well be larger but if it is there is no guarantee that the series still represents $f(z)$ at all points which are simultaneously in Ω and in the circle of convergence.

we recall that the developments

$$e^z = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

served as defns of the functions they represent.

⇒ It amounts to the same thing if we develop

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the function $(1+z)^{\mu}$ or $\log(1+z)$ about the origin choosing the branch which is respectively equal to 1 or 0 at the origin.

$\Rightarrow$ Since this branch is single valued and analytic in $|z|<1$, the radius of convergence is atleast 1.

119 $\Rightarrow$ It is elementary to compute the co-efficients and we obtain

$$(1+z)^{\mu} = 1+\mu z + \binom{\mu}{2} z^2 + \dots + \binom{\mu}{n} z^n + \dots$$

$$\Rightarrow \log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \dots$$

where the binomial co-efficients are defined by

$$\binom{\mu}{n} = \frac{\mu(\mu-1)\dots(\mu-n+1)}{1\cdot 2\dots n}$$

$\Rightarrow$ If the logarithmic series had a radius of convergence greater than 1, then $\log(1+z)$ would be bounded for $|z|<1$.

$\Rightarrow$ If Since this is not the case, the radius of convergence must be exactly 1.

$\Rightarrow$ If the binomial series were convergent in a circle of radius >1.

$\Rightarrow$ The function $(1+z)^{\mu}$ and all its derivatives would be bounded in $|z|<1$.

unless μ is a positive integer, one of the derivatives will be a negative power of $1+z$, an hence unbounded.

Thus the radius of convergence is precisel 1 except in the trivial case in which the binomi series reduces to a polynomial.

The series developments of the cyclo me functions arc tan z and arc sin z are most eas obtained by consideration of the derived series.

from the expansions

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \dots$$

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we obtain by integration

$$\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$$

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where the branch is uniquely determined as

$$\arctan z = \int_0^z \frac{dz}{1+z^2}.$$

for any path inside the unit circle

$\Rightarrow$ on uniform convergence or apply thm 1

$\Rightarrow$ The radius of convergence cannot be greater than that of the derived series and hence it is exactly 1.

$\Rightarrow$ If $\sqrt{1-z^2}$ is the branch with a positive real part, we have

$$\frac{1}{\sqrt{1-z^2}} = 1 + \frac{1}{2}z^2 + \frac{1}{2}\cdot\frac{3}{4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}z^6 + \cdots$$

for $|z|<1$ and through integration we obtain

$$\arcsin z = z + \frac{1}{2}\frac{z^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{z^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{z^7}{7} + \cdots$$

The series represents the principle branch of $\arcsin z$ with a real part between $-\pi/2$ and $\pi/2$

$\Rightarrow$ Analytic at the origin can be written in the form

$f(z) = a_0 + a_1 z + \cdots + a_n z^n + [z^{n+1}]$ where the co-efficients are uniquely determined and equal to the taylor co-efficients of f.

$\Rightarrow$ Thus in order to find the first n coefficients of the taylor expansion. It is sufficient to determine a polynomial $P_n(z)$ such that $f(z) - P_n(z)$ has a zero atleast order $n+1$ at the origin.

$\Rightarrow$ The degree of $P_n(z)$ does not matter, it is true in any case that the coefficients of z^m, $m \leq n$ are the taylor co-efficients of $f(z)$.

for instance suppose that

$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$

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$g(z) = b_0 + b_1 z + b_2 z^2 + \dots + b_n z^n + \dots$

with an abbreviated notation

we write

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$f(z) = P_n(z) + [z^{n+1}]$;

$g(z) = Q_n(z) + [z^{n+1}]$

$\Rightarrow$ It is then clear that $f(z)g(z) = P_n(z)\,Q_n(z) + [z^{n+1}]$ and the co-efficients of the term of degree $\leq n$ in $P_n Q_n$ are the taylor co-efficients of the product $f(z)g(z)$

we obtain

$$f(z)g(z) = a_0 b_0 + (a_0 b_1 + a_1 b_0) z + \dots + (a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0) z^n + \dots$$

$\Rightarrow$ If $g(z)$ is developed around z_0, the expansion of $f(\omega)$ must be in powers of $\omega - g(z_0)$

To simplify let us assume that $z_0 = 0$ and $g(0) = 0$

we can then set $f(\omega) = a_0 + a_1 \omega + \dots + a_n \omega^n + \dots$

$g(z) = b_1 z + b_2 z^2 + \dots + b_n z^n + \dots$

using the same notation as before we write

$f(\omega) = P_n(\omega) + [\omega^{n+1}]$ and

$g(z) = Q_n(z) + [z^{n+1}]$ with $Q_n(0) = 0$

$\Rightarrow$ Sub $\omega = g(z)$. we have to observe that

$$P_n(Q_n + [z^{n+1}]) = P_n(Q_n(z)) + [z^{n+1}].$$

Thus we obtain $f(g(z)) = P_n(Q_n(z)) + [z^{n+1}]$ and the taylor co-efficients of $f(g(z))$ are the co-efficients of $P_n(Q_n(z))$ for powers $\leq n$.

finally, we must be able to expand the inverse function of an analytic function $\omega = g(z)$

Here we may suppose that $g(0) = 0$.

and we are looking for the branch of the inverse functions $z = g^{-1}(\omega)$ which is analytic in a neighbourhood of the origin and vanishes for $\omega = 0$

for the existence of the inverse function it is necessary & sufficient that $g'(0) \neq 0$.

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hence we assume that

$g(z) = a_1 z + a_2 z^2 + \dots = Q_n(z) + [z^{n+1}]$.

with $a_1 \neq 0$ our problem is to determine a polynomial $P_n(w)$ such that $P_n(Q_n(z)) = z + [z^{n+1}]$

In fact under the assumption $a_1 \neq 0$ the notations $[z^{n+1}]$ and $[w^{n+1}]$ are interchangeable and from $z = P_n(Q_n(z)) + [z^{n+1}]$

we obtain

$$z = P_n(g(z) + [z^{n+1}]) + [z^{n+1}] = P_n(w) + [w^{n+1}].$$

Hence $P_n(w)$ determines the co-efficients of $g^{-1}(w)$.

In order to prove the existence of a polynomial P_n we proceed by induction

clearly we can take $P_1(w) = w/a_1$.

If P_{n-1} is given, we get $P_n = P_{n-1} + b_n w^n$ and obtain

$$P_n(Q_n(z)) = P_{n-1}(Q_n(z)) + b_n a_1^n z^n + [z^{n+1}]$$

$$= P_{n-1}(Q_{n-1}(z) + a_n z^n) + b_n a_1^n z^n + [z^{n+1}]$$

$$= P_{n-1}(Q_{n-1}(z)) + P'_{n-1}(Q_{n-1}(z)) a_n z^n + b_n a_1^n + [z^{n+1}]$$

In the last member the first two terms form a known polynomial of the form

$z + c_n z^n + [z^{n+1}]$,

and we have only to take

$b_n = -c_n a_1^{-n}$

for practical purposes the developments of the inverse function is found by successive substitutions.

To illustrate the method we determined

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the expansion of tan ω from the series.

$$\omega = \arc\tan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots$$

If we want the development to include fifth powers we write

$$z = \omega + \frac{z^3}{3} - \frac{z^5}{5} + [z^7].$$

and substitute this expression in the te to the right. with appropriate remainders we obtain

$$z = \omega + \tfrac{1}{3}(\omega + z^3/3 + [\omega^5])^3 - \tfrac{1}{5}(\omega + [\omega^3])^5 + [$$

$$= \omega + \tfrac{1}{3}\omega^3 + \tfrac{1}{3}\omega^2 z^3 - \tfrac{1}{5}\omega^5 + [\omega^7].$$

$$= \omega + \tfrac{1}{3}\omega^3 + \tfrac{1}{3}\omega^2(\omega + [\omega^3])^3 - \tfrac{1}{5}\omega^5 + [\omega^7]$$

$$= \omega + \tfrac{1}{3}\omega^3 + \tfrac{2}{15}\omega^5 + [\omega^7].$$

Thus the development of tan ω begi with the terms

$$\tan \omega = \omega + \tfrac{1}{3}\omega^3 + \tfrac{2}{15}\omega^5 + \cdots$$